Advanced General Relativity: Exercises

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Listed below are the exercises that have been assigned during the course and collected according to the lecture in which they were assigned. These exercises can be solved independently or together during the exercise time. Some of these questions could be part of the oral exam.

Lecture I

1. Prove the following relation holds for the Riemann tensor

$$3R_{\alpha[\beta\gamma\delta]} = R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} \,. \tag{1}$$

   Next, using the Bianchi identities

$$\nabla_{[\alpha} R_{\beta\delta]\mu\nu} = 0 = \nabla_{\alpha]} R_{\mu\nu[\beta\delta} \,, \tag{2}$$

   show there exists a tensor

$$G^{\alpha\beta} := R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \,, \tag{3}$$

   where $R_{\alpha\beta} := R^{\mu}_{\ \alpha\mu\beta}$ and $R := R^{\alpha}_{\ \alpha}$ are respectively the Ricci tensor and Ricci scalar and such that

$$\nabla_{\alpha} G^{\alpha\beta} = 0 \,. \tag{4}$$

2. Prove that if $\boldsymbol{\xi}$ is the separation four-vector between two neighbouring geodesics with tangent vector $\boldsymbol{u}$, then the following expression can be derived

$$\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{u}} \xi^{\alpha} = -R^{\alpha}_{\ \beta\mu\nu} u^{\beta} \xi^{\mu} u^{\nu} \,. \tag{5}$$

   This is known as the geodesic-deviation equation.

3. ***Optional.*** Consider the Einstein equations written generically as a tensor expression equating geometry and energy, *i.e.*,

$$G_{\alpha\beta} + \kappa_1 \Lambda g_{\alpha\beta} = \kappa_2 T_{\alpha\beta} \,, \tag{6}$$

   where κ_1 and κ_2 are two generic constant and Λ the cosmological constant. Show that the matching with the Newtonian limit yields

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} \,. \tag{7}$$

Lecture II

1. Show that when using the Lagrangian

$$2L = \mathcal{L}^2 = g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \,, \tag{8}$$

   where $\dot{x}^{\mu} = dx^{\mu}/d\lambda$ and λ is the affine parameter of a massive particle, the *Euler–Lagrange equations*

$$\frac{\partial L}{\partial x^{\mu}} - \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^{\mu}} \right) = 0 \,, \tag{9}$$

yield the geodesic equations

$$\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \,. \tag{10}$$

2. Using as affine parameter $\lambda = \tau/m$, where m is the mass of the particle, show that the Lagrangian (8) relative to a Schwarzschild spacetime and the Euler–Lagrange equations yield the following geodesic equations

$$\frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}\right] = 0 \,, \tag{11}$$

$$\begin{aligned}\frac{d}{d\tau}\left[\left(1-\frac{2M}{r}\right)^{-1}\frac{dr}{d\tau}\right] = \; & r\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right] \\ & - \frac{M}{r^2}\left[\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-2}\left(\frac{dr}{d\tau}\right)^2\right] \,,\end{aligned} \tag{12}$$

$$\frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right) = r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 \,, \tag{13}$$

$$\frac{d}{d\tau}\left(r^2\sin^2\theta\frac{d\phi}{d\tau}\right) = 0 \,. \tag{14}$$

3. Show that the constant of motions given by the covariant components of a particle's four-momentum p_t and p_ϕ represent the energy at infinity and the specific angular momentum, respectively.

4. ***Optional.*** Show that the lowest specific angular momentum allowing for the existence of circular orbits is $\tilde{\ell}^2 := (\ell/m)^2 = 12M^2$. Calculate the radii of the corresponding stable and unstable circular orbits.

Lecture III

1. Derive the values of the specific angular momentum and specific energy for a massive particle in circular orbit around a Schwarzschild spacetime.

2. Derive the values of the specific angular momentum relative to the marginally bound orbit for a massive particle in a Schwarzschild black hole.

3. ***Optional.*** Calculate the expression of the tetrad components carried by a Zero Angular Momentum Observer (ZAMO).

Lecture IV

1. Show that the frame-dragging angular in a Kerr spacetime is given by the expression
$$\omega(r,\theta) := \frac{p^\phi}{p^t} = \frac{2Mra}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}\,. \tag{15}$$
Further show that the angular velocity of a massive particle is constrained to be
$$\Omega_{\rm min} \leq \Omega \leq \Omega_{\rm max}\,, \tag{16}$$
where
$$\Omega_{\rm min,max} := \omega \mp \sqrt{\omega^2 - g_{00}/g_{\phi\phi}}\,. \tag{17}$$
Plot these two angular velocities for different values of the spin parameter. What is the angular velocity of the horizon?

2. Consider a photon emitted by a static observer in a Schwarzschild spacetime and and propagating in the direction $\boldsymbol{k}$. Let ψ be the angle between the direction of propagation of the photon and the unit radial four-vector of the tetrad carried by the static observer. Compute at what angles an *ingoing photon* should be fired to reach infinity if the observer is at $r = 6M, r = 3M$ and $r = 2M$. Repeat the considerations for an *outgoing photon*. Draw a sketch to illustrate the behaviour.

3. ***Optional.*** Let $E_{\rm ZAMO}$ be the energy of a particle measured by a Zero Angular Momentum Observer (ZAMO) in a Kerr spacetime. Show that
$$E_{\rm ZAMO} = A(E - \ell\omega)\,, \tag{18}$$
where
$$A := \frac{g_{\phi\phi}}{g_{0\phi}^2 - g_{00}g_{\phi\phi}} > 0\,, \tag{19}$$
and E, ℓ are the energy, angular momentum of the particle and ω is the frame-dragging angular velocity

Lecture V

1. Show that the specific energy and specific angular momentum for circular orbits of massive particles in a Kerr spacetime are given by
$$\tilde{E} = \frac{r^2 - 2Mr \pm a\sqrt{Mr}}{r\left(r^2 - 3Mr \pm 2a\sqrt{Mr}\right)^{1/2}}\,,$$
$$\tilde{\ell} = \pm\frac{\sqrt{Mr}\left(r^2 \mp 2a\sqrt{Mr} + a^2\right)}{r\left(r^2 - 3Mr \pm 2a\sqrt{Mr}\right)^{1/2}}\,,$$
where the $\pm$ signs refers to corotating and counterrotating particles, respectively.

2. Compute the energy drop for a particle that is at rest at spatial infinity and spirals down to the ISCO in a Kerr spacetime. Evaluate this quantity as a function of the black-hole spin and estimate the largest value.

3. ***Optional.*** Compute the expression for the Keplerian angular velocity for a massive particle in a Kerr spacetime and the corresponding specific angular momentum.

Lecture VI

1. If $A_{\rm BH}$ is the area of a Kerr black hole of mass M and spin $a = J/M$, show that the requirement of the increase in the area can still be satisfied by transformations in which both the mass and the spin of the black hole decrease, *i.e.*,

$$\delta A_{\rm BH} > 0 \qquad \Longleftrightarrow \qquad \frac{M\,\delta M}{a\,\delta a} > 1\,.$$

2. Derive the TOV equations describing equilibrium configurations of relativistic static and spherically symmetric stars

$$\frac{dm}{dr} = 4\pi r^2 e\,, \tag{20}$$

$$-(e+p)\frac{d\phi}{dr} = -\frac{(e+p)(m+4\pi r^3 p)}{r(r-2m)} = \frac{dp}{dr}\,, \tag{21}$$

3. ***Optional.*** Show that for a perfect but anisotropic fluid with energy-momentum tensor in the comoving frame given by

$$(T_{\hat{\mu}\hat{\nu}})_{\rm anisotropic} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{pmatrix}\,,$$

the hydrostatic-equilibrium equation is changed to

$$\frac{dp_r}{dr} = -\frac{(e+p_r)(m+4\pi r^3 p_r)}{r(r-2m)} + \frac{2(p_t - p_r)}{r}\,,$$

where p_r, p_t are the radial and tangential pressures, respectively. Explain how the tangential pressure is computed.

Lecture VII

No exercises for this lecture to allow people to catch-up with unsolved exercises.

Lecture VIII-IX

1. Starting from from the generic diagonal line element in spherical symmetry written in the form

$$ds^2 = -a(r,t)^2 dt^2 + b(r,t)^2 dr^2 + R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) , \qquad (22)$$

re-derive equations (28)-(32) appearing on pages 122-123 of the lecture notes. These equations are also known as the "Misner-Sharp" equations and they represent the simplest formulation of the Einstein equations in spherical symmetry.

2. Show that under the assumptions that the fluid is homogeneous but not pressureless ($D_r p = 0$, $p \neq 0$), the Misner-Sharp equations lead to the Friedmann equations

$$\ddot{S} + \frac{4\pi}{3}(e + 3p)S = 0 \, ,$$
$$\dot{S}^2 - \frac{8\pi}{3} e S^2 = -\kappa \, ,$$

where S and κ are the conformal spatial factor and the curvature constant of the Friedmann-Robertson-Walker line element

$$ds^2 = -dt^2 + S^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2\right) . \qquad (23)$$

3. If $\mathcal{A}$ is the area of a swarm of radially outgoing photons inside a collapsing dust cloud (OS collapse), show that

$$\frac{d\mathcal{A}}{d\eta} \leq 0 \, , \qquad (24)$$

is equivalent to the condition

$$\eta_e \geq \pi - 2\chi_e \, , \qquad (25)$$

where η_e and χ_e are the time and position of emission and where we have written the interior line element as

$$ds^2 = -d\tau^2 + S^2 \left[d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] . \qquad (26)$$

4. ***Optional.*** Use the following definitions

$$\mathcal{C}_e := \int_0^{2\pi} \sqrt{g_{\phi\phi}} d\phi \, , \qquad \mathcal{C}_p := 2\int_0^{\pi} \sqrt{g_{\theta\theta}} d\theta \, , \qquad (27)$$

for the equatorial and polar proper circumferences of the event horizon of a Kerr black hole. Show that $\mathcal{C}_p/\mathcal{C}_e = 1$ for $J/M^2 = 0$, but that $\mathcal{C}_p/\mathcal{C}_e \neq 1$ for $J/M^2 \neq 0$. Derive the generic expression for $\mathcal{C}_p/\mathcal{C}_e$ and compute it in the case of an extremal Kerr black hole ($J = M^2$).

Lecture X

1. Prove that if $\boldsymbol{u}$ is a timelike unit four-velocity (*i.e.*, $u^\mu u_\mu = -1$), the covariant three-velocity defined as

$$v^i = -\frac{\gamma^i{}_\mu u^\mu}{n_\mu u^\mu} \,, \tag{28}$$

has components given by

$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right) \,. \tag{29}$$

Also prove that the quantity $W := \alpha u^t$ is the Lorentz factor since it satisfies the identity

$$W = \left(1 - v^i v_i\right)^{-1/2} \,. \tag{30}$$

Compare expression (29) with the equivalent expression in special relativity.

2. Recalling that the Schwarzschild solution in quasi-isotropic coordinates reads

$$ds^2 = -\left(\frac{1 - M/(2r)}{1 + M/(2r)}\right) dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right) \,, \tag{31}$$

compute the components of the one-form $\boldsymbol{\Omega}$, of the unit normal $\boldsymbol{n}$, of the lapse function α, of the shift vector $\boldsymbol{\beta}$, and of the three metric $\boldsymbol{\gamma}$; for all tensors compute both the covariant and the contravariant components.

3. ***Optional.*** Consider a metric in a 3+1 split and the geodesic equation for a massless particle with four momentum $\boldsymbol{p}$, such that $\boldsymbol{p} \cdot \boldsymbol{p} = 0$. Show that the geodesic equation can be written as

$$\begin{aligned} \frac{dx^j}{d\lambda} &= \gamma^{ij} p_i - \beta^j p^0 \,, \\ \frac{dp_i}{d\lambda} &= -(p^0)^2 \alpha \partial_i \alpha + p^0 p_j \partial_i \beta^j - \frac{1}{2} p_l p_m \partial_i \gamma^{lm} \,, \end{aligned}$$

where λ is the affine parameter. [Hint: it is useful to prove first the following identity: $p^0 = \sqrt{\gamma^{ij} p_i p_j}/\alpha$.]

Lecture XI

1. Within a 3+1 split of spacetime, prove the following expression for the extrinsic curvature $\boldsymbol{K}$

$$\mathscr{L}_{\boldsymbol{n}} \gamma_{\mu\nu} = -2K_{\mu\nu} \,, \tag{32}$$

where $\boldsymbol{\gamma}$ is the metric associated to Σ_t and $\boldsymbol{n}$ the corresponding unit normal.

2. Prove that $a_\nu = D_\nu \ln \alpha$.

3. Derive the Gauss–Codazzi equations

$$\gamma^\mu{}_\alpha \, \gamma^\nu{}_\beta \, \gamma^\rho{}_\delta \, \gamma^\sigma{}_\lambda \, R_{\mu\nu\rho\sigma} = {}^{(3)}R_{\alpha\beta\delta\lambda} + K_{\alpha\delta}K_{\beta\lambda} - K_{\alpha\lambda}K_{\beta\delta} \,. \tag{33}$$

4. ***Optional.*** Derive the Codazzi-Mainardi equations

$$\gamma^\rho{}_\beta \gamma^\mu{}_\alpha \gamma^\nu{}_\lambda n^\sigma R_{\rho\mu\nu\sigma} = D_\alpha K_{\beta\lambda} - D_\beta K_{\alpha\lambda} \,. \tag{34}$$

If you are still having fun, derive the Ricci equations

$$\gamma^\alpha{}_\mu \gamma^\beta{}_\nu n^\delta n^\lambda R_{\alpha\delta\beta\lambda} = \mathscr{L}_{\boldsymbol{n}} K_{\mu\nu} - \frac{1}{\alpha} D_\mu D_\nu \alpha + K^\lambda{}_\nu \, K_{\mu\lambda} \,. \tag{35}$$

Lecture XII

1. Derive the form of the Hamiltonian and momentum constraints in a conformal and traceless formulation of the Einstein equations.

2. Given the conformal transformation $\tilde{\gamma}_{ij} = \phi^2 \gamma_{ij}$, $\tilde{\gamma}^{ij} = \phi^{-2}\gamma^{ij}$, show that the second covariant derivative of the conformal factor is given by

$$\tilde{D}_i \tilde{D}_j \phi = \partial_i \partial_j \phi - \tilde{\Gamma}^k_{ij} \, \partial_k \phi \,,$$

and that the relation with the corresponding derivative in the physical metric is given by

$$D_i D_j \phi = \tilde{D}_i \tilde{D}_j \phi + \frac{2}{\phi} \partial_i \phi \, \partial_j \phi - \frac{1}{\phi} \gamma_{ij} \partial^k \phi \, \partial_k \phi \,.$$

3. ***Optional.*** Show that the following definitions of energy-momentum tensor are equivalent

$$T^{\mu\nu} = (e+p) u^\mu u^\nu + p g^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu} \,,$$
$$T^{\mu\nu} = E n^\mu n^\nu + S^\mu n^\nu + S^\nu n^\mu + S^{\mu\nu} \,.$$

where E, S^μ and $S^{\mu\nu}$ are the Eulerian energy density, the momentum density and the purely spatial energy-momentum tensor, respectively. Show also that the following definitions are possible for these quantities

$$S^{\mu\nu} = \rho h W^2 v^\mu v^\nu + p \gamma^{\mu\nu} \,,$$
$$S^\mu = \rho h W^2 v^\mu \,,$$
$$E = \rho h W^2 - p \,.$$

Lecture XIII

1. Let V be the volume enclosed by the three-dimensional surface Σ on a spacelike hypersurface, so that the (proper) volume element is given by

$$V = \int_\Sigma \sqrt{\gamma}\, d^3x\,, \tag{36}$$

where, as usual, $\gamma = \det(\gamma_{ij})$. The volume V can be thought as the volume delimited by a closed two-dimensional surface $\mathscr{S}$, where $\mathscr{S}$ is of course part of a two-dimensional surface, say, Σ_0. Show that the variation of V in time when $\mathscr{S}$ remains fixed is given by

$$\partial_t V = -\int_\Sigma \alpha K \sqrt{\gamma}\, d^3x\,, \tag{37}$$

where, as usual, α is the lapse and K is the trace of the extrinsic curvature. This expression shows that a slicing with $K = 0$ is "maximal" in the sense that the volume V is an extremal with respect to variations of the domain enclosed by $\mathscr{S}$.

2. The "harmonic slicing" is the slicing requiring that the "harmonic condition"

$$\Box\, x^\alpha = \nabla_\mu \nabla^\mu\, x^\alpha = 0\,. \tag{38}$$

holds only for the time coordinate $x^0 = t$, *i.e.*, that

$$\Box\, t = 0\,. \tag{39}$$

Show that this condition corresponds to the following prescription for the lapse

$$\left(\partial_t - \beta^i \partial_i \beta\right) \alpha = -\alpha^2 K\,. \tag{40}$$

3. ***Optional.*** The "minimal distortion condition" imposes that

$$D^j \Sigma_{ij} = 0\,, \tag{41}$$

where[1]

$$\Sigma_{ij} = \Theta_{ij} - \frac{1}{3}\gamma_{ij}\Theta = \Theta_{ij} - \frac{1}{3}\gamma_{ij}\Theta_{kl}\gamma^{kl} = \frac{1}{2}\gamma^{1/3}\mathscr{L}_{\boldsymbol{t}}\tilde{\gamma}_{ij}\,, \tag{42}$$

is the metric distortion tensor and

$$\Theta_{ij} = \frac{1}{2}\mathscr{L}_{\boldsymbol{t}}\gamma_{ij} = \frac{1}{2}\partial_t \gamma_{ij}\,, \tag{43}$$

is the metric strain tensor. The "Gamma-freezing" shift condition, on the other hand, imposes that

$$\partial_t \tilde{\Gamma}^i = 0\,. \tag{44}$$

[1] Note that Eq. (7.132) of my book contains two (!) typos. The one reported here is the correct expression.

Show that

$$\partial_t \tilde{\Gamma}^i = 2\phi^{-2} \left[D_j \Sigma^{ij} - \tilde{\Gamma}^i{}_{jk} \Sigma^{jk} - 6\Sigma^{ij} \partial_j \phi \right] . \tag{45}$$

and hence that the Gamma-freezing shift condition and the minimal-distortion condition are equivalent up to terms involving the conformal factor and its derivatives.

Lecture XIV

1. Considering the physical and conformal three metrics $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$ with $D_i \gamma_{jk} = 0 = \bar{D}_i \bar{\gamma}_{jk}$, prove that the corresponding Ricci scalars $R, \bar{R}$ associated with the original, and the conformal 3-geometry are related according to the following identity

$$R \;=\; \psi^{-4} \bar{R} - \frac{8}{\psi^5} \bar{D}_i \bar{D}^i \psi \,.$$

2. Prove that for any symmetric trace-free tensor U^{ij} we have

$$D_j U^{ij} \;=\; \psi^{-n} \bar{D}_j (\psi^n U^{ij}) + (10 - n) U^{ij} \bar{D}_j \ln \psi \,.$$

3. Given the *longitudinal operator* $\boldsymbol{L}$ defined as

$$(\boldsymbol{L}W)^{ij} \coloneqq 2D^{(i} W^{j)} - \frac{2}{3} \gamma^{ij} D_k W^k \,.$$

where W^i is a generic vector, show that the following identity holds

$$(\boldsymbol{L}\beta)^{ij} \;=\; \psi^{-4} (\bar{\boldsymbol{L}}\beta)^{ij} \,.$$

Similarly, show that for $\bar{\beta}_i = \psi^{-4} \beta_i$ we have

$$(\boldsymbol{L}\beta)_{ij} \;=\; \psi^4 (\bar{\boldsymbol{L}}\bar{\beta})_{ij} \,.$$