SoSe 2015 Advanced General Relativity	Lecture 1 - 26/04/2014

Advanced General Relativity, Tutorial

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Lecture I

Exercise 1

Using the antisymmetric property of the Riemann tensor we can simply write down

$$3R_{\alpha[\beta\gamma\delta]} = \frac{3}{3!}\left(R_{\alpha\beta\gamma\delta} - R_{\alpha\beta\delta\gamma} + R_{\alpha\gamma\delta\beta} - R_{\alpha\gamma\beta\delta} + R_{\alpha\delta\beta\gamma} - R_{\alpha\delta\gamma\beta}\right) \tag{1}$$

$$= R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma}. \quad \square \tag{2}$$

Using the properties of the Riemann tensor we can rewrite the Bianchi identity as

$$\nabla_{\lambda]} R_{\alpha\beta[\gamma\nu} \equiv R_{\alpha\beta\gamma\nu;\,\nu} + R_{\alpha\beta\lambda\gamma;\,\nu} + R_{\alpha\beta\nu\lambda;\,\mu}. \tag{3}$$

To construct the Einstein tensor from the Bianchi identities we apply the Ricci contraction

$$g^{\alpha\mu}\left[R_{\alpha\beta\mu\nu;\,\lambda} + R_{\alpha\beta\lambda\mu;\,\nu} + R_{\alpha\beta\nu\lambda;\,\mu}\right] = 0. \tag{4}$$

Given the property of the covariant derivative applied to the metric $g^{\alpha\beta}_{;\mu} = 0$ we can write down

$$R_{\alpha\nu;\,\lambda} + (-R_{\beta\lambda;\,\nu}) + R^{\mu}_{\beta\nu\lambda;\,\mu} = 0. \tag{5}$$

And contracting again on the indices ν and β

$$R_{;\,\lambda} - R^{\mu}_{\lambda;\mu} + \left(-R^{\mu}_{\lambda;\,\mu}\right) = 0. \tag{6}$$

Using the definition of the Ricci scalar R

$$(2R^{\mu}_{\lambda} - \delta^{\mu}_{\lambda}R)_{;\,\mu} = 0, \tag{7}$$

and introducing a symmetric tensor $G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R$, we can see that Eq. (7) is equivalent to

$$\nabla^{\alpha}G_{\alpha\beta} = 0. \tag{8}$$

Exercise 2

Given a connecting 4-vector $\underline{\xi}$ between two neighbouring geodesics, we say that $\underline{\xi}$ is Lie propagated if its derivative with respect to $\underline{u}$ vanishes

$$\mathcal{L}_{\underline{u}}\,\underline{\xi} = u^{\beta}\nabla_{\beta}\xi^{\alpha} - \xi^{\beta}\nabla_{\beta}u^{\alpha} = 0. \tag{9}$$

since the Lie derivative vanishes, we can write

$$u^{\beta}\nabla_{\beta}\xi^{\alpha} = \xi^{\beta}\nabla_{\beta}u^{\alpha}. \tag{10}$$

In analogy with the Newtonian expression for the geodesic deviation derived during the lecture, the relative acceleration between of the 4-vector $\underline{\xi}$ is given by

$$\nabla_{\underline{u}}\nabla_{\underline{u}}\xi^{\alpha} = u^{\beta}\nabla_{\beta}(u^{\gamma}\nabla_{\gamma}\xi^{\alpha}) \tag{11}$$

$$= u^{\beta}\nabla_{\beta}\xi^{\gamma}\nabla_{\gamma}u^{\alpha} + u^{\beta}\xi^{\gamma}\nabla_{\beta}\nabla_{\gamma}u^{\alpha}. \tag{12}$$

We can use the expression $\nabla_{\beta}\nabla_{\gamma}u^{\alpha} - \nabla_{\gamma}\nabla_{\beta}u^{\alpha} = R^{\alpha}{}_{\delta\beta\gamma}u^{\delta}$ (see lecture notes)

$$\nabla_{\underline{u}}\nabla_{\underline{u}}\xi^{\alpha} = u^{\beta}\nabla_{\beta}\xi^{\gamma}\nabla_{\gamma}u^{\alpha} + \xi^{\gamma}u^{\beta}\nabla_{\gamma}\nabla_{\beta}u^{\alpha} + \xi^{\gamma}u^{\beta}u^{\delta}R^{\alpha}{}_{\delta\beta\gamma}. \tag{13}$$

And applying the Leibniz rule and some index relabelling we obtain

$$\nabla_{\underline{u}}\nabla_{\underline{u}}\xi^{\alpha} = \xi^{\beta}\left[\nabla_{\beta}(u^{\gamma}\nabla_{\gamma}u^{\alpha})\right] + \xi^{\gamma}u^{\beta}u^{\delta}R^{\alpha}{}_{\delta\beta\gamma}, \tag{14}$$

where the term $u^{\gamma}\nabla_{\gamma}u^{\alpha} = 0$ (tangent vector to the geodesic). After rearranging the indices we arrive at the equation for the geodesic deviation

$$\nabla_{\underline{u}}\nabla_{\underline{u}}\xi^{\alpha} = R^{\alpha}{}_{\beta\gamma\delta}u^{\beta}u^{\gamma}\xi^{\delta}, \tag{15}$$

the sign change in Eq. (5) on the exercise sheet is again a consequence of the properties of Riemann tensor.

Exercise 3

We start by contracting the Einstein's field equations with a cosmological constant Λ using $g^{\alpha\beta}$,

$$R - 2R + 4\Lambda = 8\pi T \implies R = -8\pi T + 4\Lambda, \tag{16}$$

We can use this result to rewrite the Einstein's equations as

$$R_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T) + \Lambda g_{\alpha\beta}. \tag{17}$$

If we plug back this expression into Eq. (6) of the exercise sheet, we obtain

$$R = \kappa_2 T + 2R - 4\kappa_1 \implies R = 4\kappa_1 - \kappa_2 T, \tag{18}$$

similarly

$$R_{\alpha\beta} = \kappa_2(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T) + \kappa_1 g_{\alpha\beta}. \tag{19}$$

In the weak-field limit the componentes of the stress energy-tensor $T^{00} \gg T^{0j} \gg T^{ij}$, the 00 covariant component of the previous expression becomes

$$R_{00} = \kappa_2(T_{00} - \frac{1}{2}g_{00}T) - g_{00}\Lambda' \tag{20}$$

where we absorbed κ_1 in the definition of the cosmological constant. With a bit of manipulation we can derive

$$\kappa_2(T_{00} - \frac{1}{2}g_{00}T) = T_{00} \approx \eta_{0\beta}\eta_{0\alpha}\frac{T^{00}}{2} = \kappa_2\frac{\rho c^2}{2}. \tag{21}$$

Eq. (20) becomes

$$R_{00} = \kappa_2\frac{\rho c^2}{2} + g_{00}\Lambda', \tag{22}$$

the left-hand side can be computed in the weak field regime as

$$R_{00} = -\frac{1}{2}h^{i}_{00,i} = -\frac{1}{2}\nabla^2 h_{00}; \tag{23}$$

where $h_{\mu\nu}$ is a correction to the flat space-time metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We need to relate h_{00} to the Newtonian potential ϕ and we can do so by using the geodesic equation for a freely falling particle

$$\frac{{}^2x^\mu}{d\tau^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0, \tag{24}$$

and the equation of motion in the presence of a gravitational field $\ddot{\vec{x}} = -\vec{\nabla}\phi$. Considering only the spatial components and neglecting terms of $O(h^2)$

$$\Gamma^{\mu}_{\alpha\beta}u^\alpha u^\beta \approx \Gamma^{\mu}_{0}0^{0}_{u}u^0 = \Gamma^{\mu}_{00}, \tag{25}$$

and by comparing with the Newtonian equation of motion we obtain

$$\ddot{\vec{x}} = -\Gamma^{\mu}_{00} = -\frac{1}{2}\nabla^2 h_{00}; \tag{26}$$

from which we can easily derive $h_{00} = -2\frac{\phi}{c^2}$. If we go back to the Einstein equations,

$$R_{00} = -\frac{1}{2}\nabla^2(-\frac{2\phi}{c^2}) = \frac{\nabla^2\phi}{c^2}, \tag{27}$$

and putting all together with $\nabla^2\phi = 4\pi G\rho$, we can compute the value of κ

$$\frac{\nabla^2\phi}{c^2} = \frac{\kappa\rho c^2}{2} + g_{00}\Lambda\ . \tag{28}$$

The Poisson equation is

$$\nabla^2\phi = 4\pi G(\rho + \rho_\lambda)\ , \tag{29}$$

so that

$$\frac{\nabla^2\phi}{c^2} = \frac{4\pi G}{c^2}(\rho + \rho_\lambda) = \frac{\kappa\rho c^2}{2} + \Lambda\ , \tag{30}$$

where ρ_λ is the "mass-density" of vacuum. From this expression we can easily derive the expressions for κ and Λ

$$\kappa = \frac{8\pi G}{c^4}\ , \tag{31}$$

$$\rho_\lambda = \frac{\Lambda c^2}{4\pi G}\ . \tag{32}$$

$\square$.

Lecture II

Exercise 1

$$\frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0, \tag{33}$$

where $\dot{x}^\mu = \frac{dx^\mu(\lambda(s))}{ds}$. Since

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = 2g_{\nu\mu}\dot{x}^\nu \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial x^\mu} = g_{\nu\kappa,\mu}\dot{x}^\mu\dot{x}^\kappa, \tag{34}$$

the *Euler-Lagrange* equations become

$$g_{\nu\mu}\ddot{x}^\nu + (g_{\nu\mu,\kappa} - \frac{1}{2}g_{\nu\kappa,\mu})\dot{x}^\nu\dot{x}^\kappa = 0. \tag{35}$$

We can simply rewrite the previous equation in the following form

$$g_{\nu\mu}\ddot{x}^\nu + (g_{\nu\mu,\kappa} + g_{\kappa\mu,\nu} - g_{\nu\kappa,\mu})\dot{x}^\nu\dot{x}^\kappa = 0, \tag{36}$$

and contracting with $g^{\nu\alpha}$ it reduces to well know geodesic equation

$$\ddot{x}^\alpha + \Gamma^\alpha_{\nu\kappa}\dot{x}^\nu\dot{x}^\kappa = 0. \tag{37}$$

Exercise 2

Given the Schwarzschild metric, we can write the Lagrangian for the system as

$$\mathcal{L} = \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - 2M/r} - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}\right], \tag{38}$$

where the dot represents the derivative respect to τ. We can calculate the respective canonical momenta

$$p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = \left(1 - \frac{2M}{r}\right)\dot{t}, \tag{39}$$

$$p_\phi = -\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = (r^2\sin^2\theta)\dot{\phi}, \tag{40}$$

$$p_r = -\frac{\partial \mathcal{L}}{\partial \dot{r}} = \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}, \tag{41}$$

$$p_\theta = -\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2\dot{\theta}. \tag{42}$$

From the equation of motion for the θ coordinate

$$\frac{dp_\theta}{d\tau} = \frac{d}{d\tau}(r^2\dot{\theta}) = -\frac{\partial \mathcal{L}}{\partial \theta} = r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2. \tag{43}$$

We are left with the radial geodesic

$$\frac{dp_r}{d\tau} = \frac{d}{d\tau}\left[\left(1 - \frac{2M}{r}\right)^{-1}\dot{r}\right] = -\frac{\partial \mathcal{L}}{\partial r} = r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - \frac{M}{r^2}\left[\dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-1}2M\dot{r}^2\right]. \tag{44}$$

TODO: add the corrected version and remove the old one

Exercise 3

We introduce an orthonormal tetrad $\underline{e}^{\hat{\alpha}}_\mu$ such that the scalar product of the basis vectors constitute the flat metric $\eta_{\hat{\alpha}\hat{\beta}}$

$$\underline{e}_{\hat{\alpha}} \cdot \underline{e}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}\ , \tag{45}$$

$$\tilde{\omega}^{\hat{\alpha}} \cdot \tilde{\omega}^{\hat{\alpha}} = \eta^{\hat{\alpha}\hat{\beta}}\ . \tag{46}$$

The components of the tetrads are

$$\underline{e}_{\hat{t}} = \left(1 - \frac{2M}{r}\right)^{-1/2} \underline{e}_t \tag{47}$$

$$\underline{e}_{\hat{r}} = \left(1 - \frac{2M}{r}\right)^{1/2} \underline{e}_r \tag{48}$$

$$\underline{e}_{\hat{\theta}} = \frac{1}{r}\underline{e}_\theta \tag{49}$$

$$\underline{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\underline{e}_\phi \,. \tag{50}$$

The energy of a particle of mass m measured by an observer with 4-velocity $\underline{u}$ is given by

$$\begin{aligned} E_{\text{local}} - \underline{p} \cdot \underline{u} &= -p_\alpha u^\alpha && (51)\\ &= -p_\alpha \underline{e}_{\hat{t}} && (52)\\ &= -p_\alpha \left(1 - \frac{2M}{r}\right)^{-1/2} \underline{e}_t && (53)\\ &= -p_t \left(1 - \frac{2M}{r}\right)^{-1/2} && (54)\\ &= E \left(1 - \frac{2M}{r}\right)^{-1/2} \,, && (55) \end{aligned}$$

where we have used the expression for the 4-momenta $p_t = -\left(1 - \frac{2M}{r}\right)\dot{t} = -E$. Similarly for the angular velocity measured by a locally static observer

$$\begin{aligned} v^{\hat{\phi}} = \frac{p^{\hat{\phi}}}{p^{\hat{t}}} &= \frac{p_\alpha e^\alpha_{\hat{\phi}}}{p_t e^\alpha_{\hat{t}}} && (56)\\ &= \frac{p_\phi/(r\sin\theta}{E_{\text{local}}} = \frac{l/(r\sin\theta)}{E_{\text{local}}} \,, && (57) \end{aligned}$$

and we can find the requested expression $l = v^{\hat{\phi}} r \sin\theta \, E_{\text{local}}$.

Exercise 4

Combining Eq. (74) and Eq. (75) and restricting $\theta = \pi/2$, a particle

$$-\left(1 - \frac{2M}{r}\right)^{-1} E^2 + \left(1 - \frac{2M}{r}\right)^{-1} (\frac{dr}{d\tau} + \frac{l^2}{r^2} = 1 \,, \tag{58}$$

if we multiply it by $1 - \frac{2M}{r}$ and dividing by 2, we obtain

$$-\frac{e^2}{2} + \frac{1}{2}\frac{dr}{d\tau} + \frac{1}{2}\left(1 - \frac{2M}{r}\right)\frac{l^2}{r^2} = \frac{1}{2} - \frac{M}{r} \,, \tag{59}$$

isolating the energy per unit mass

$$\frac{e^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{\tilde{l}^2}{2r^2}\left(1 - \frac{2M}{r}\right) - \frac{M}{r} \,, \tag{60}$$

we can define an effective potential

$$V_{\text{eff}} = -\frac{\tilde{l}^2}{2r^2}\left(1 - \frac{2M}{r}\right) + \frac{M}{r} \,. \tag{61}$$

The first derivative of the potential

$$\frac{dV_{\text{eff}}}{dr} = \frac{d}{dr}\left(\frac{M}{r} - \frac{l^2}{2r^2} + \frac{l^2 M}{r^3}\right) = -\frac{M}{r^2} + \frac{l^2}{r^3} - \frac{3L^2 M}{r^4}\,, \tag{62}$$

in order to find the circular orbits we compute the zeros of the previous function

$$Mr^2 - l^2 r + 3l^2 m = , = 0 \tag{63}$$

we find the minimum and the maximum of the radius

$$r_{1,2} = \frac{l^2 \pm \sqrt{l^4 - 12 l^2 M^2}}{2M} = \frac{l^2}{2M}\left(1 \pm \sqrt{1 - \frac{12M^2}{l^2}}\right)\,, \tag{64}$$

which gives us the condition for the existence of a circular orbit $l^2 < 12M^2$. At the minimum value of the potential we can find a stable circular orbit

$$r_1 = \frac{l^2}{2M}\left(1 + \sqrt{1 - \frac{12M^2}{l^2}}\right)\,, \tag{65}$$

and obvioussly the unstable ones has a radius of

$$r_2 = \frac{l^2}{2M}\left(1 - \sqrt{1 - \frac{12M^2}{l^2}}\right)\,, \tag{66}$$

for the minimum $L/M < 2\sqrt{3}$ and so $r_{\text{c}} = 6M$. So

$$6M < r_c(\text{stable}) < +\infty\,, \tag{67}$$

$$3M \leq r_c(\text{unstable}) \leq 6M\,. \tag{68}$$

Lecture III

Exercise 1

Given the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{r^2}{1 - 2M/r} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi, \tag{69}$$

we see that there are no terms that explicitly depend on the time t and angular variable ϕ. The conjugate momenta are respectively

$$-p_t = -\frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\dot{t}\,, \tag{70}$$

$$p_\phi = -\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2 \dot{\phi} \sin^2\theta. \tag{71}$$

We can define therefore two killing vectors

$$\xi = (1,0,0,0)\,, \tag{72}$$
$$\eta = (0,0,0,1). \tag{73}$$

The related conserved quantities are found taking the dot product of each vector with the four velocity.

$$e = \xi \cdot u = g_{\alpha\beta}\xi^{\alpha}u^{\beta} = -\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}\,, \tag{74}$$
$$l = \eta \cdot u = g_{\alpha\beta}\eta^{\alpha}u^{\beta} = r^2 \sin\theta \frac{+d\phi}{d\tau} = r^2\frac{d\phi}{d\tau}. \tag{75}$$

given $\theta = \pi/2$. For a massive particle

$$\mathcal{L} = -\frac{m^2}{2}\,, \tag{76}$$

which can be easily proved

$$p_{\alpha} = -\frac{\partial \mathcal{L}}{\partial \dot{x^{\alpha}}} = g_{\alpha\beta}\dot{x^{\beta}}\,, \tag{77}$$
$$p^{\alpha} = g^{\alpha\beta}p_{\beta} = g^{\alpha\beta}g_{\beta\mu}\dot{x^{\mu}} = \dot{x}^{\alpha}. \tag{78}$$

by taking the dot product

$$p^{\alpha}_{\alpha} = g_{\alpha\beta}\dot{x^{\alpha}}\dot{x^{\beta}} = 2\mathcal{L}, p^{\alpha}_{\alpha} = -m^2, \tag{79}$$

which gives us $2\mathcal{L} = -m^2$.

Exercise 2

A massive particle in a marginally bound orbit should satisfy the following equation

$$u_t = -1 \tag{80}$$

which is equivalent to $u_t^2 = 1$ and can be written down as

$$u_t^2 = \left(1-\frac{2M}{r}\right)\left[1+\frac{Mr^2/(r-3M)}{r^2}\right] = \frac{(r-2M)^2}{r(r-3M)} \tag{81}$$
$$\tag{82}$$

which gives a radius for a marginally stable orbit of $r = r_{\mathrm{mb}} = 4M$. Given the effective potential

$$V_{\mathrm{eff}} = \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{l}}{r^2}\right)\,, \tag{83}$$

the maximum of the potential can be found by

$$\frac{\partial V_{\mathrm{eff}}}{\partial r} = r\tilde{l}^2 - 2Mr^2 - 2M\tilde{l}^2 = 0\,, \tag{84}$$

which for $r_{\mathrm{mb}} = 4M$ becomes $\tilde{l} = 4M$.

Exercise 3

Given the following line element

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(dR^2 + dz^2) \; , \tag{85}$$

all the observers whose velocity has at each point the form (for circular, axisymmetric motion)

$$u^\alpha = u^t(t^\alpha + \omega\phi^\alpha) = e^\nu(t^\alpha + \omega\phi^\alpha) \; , \tag{86}$$

represent a family of zero-angular-momentum-observers (ZAMOs). A natural tetrad in Kerr is given by the frame of a ZAMO, the basis covectors have the following expression

$$\omega^{\hat{0}} = e^\nu dt \; , \tag{87}$$
$$\omega^{\hat{1}} = e^\psi(d\phi - \omega dt) \; , \tag{88}$$
$$\omega^{\hat{2}} = e^\mu dR \; , \tag{89}$$
$$\omega^{\hat{3}} = e^\mu dz \; , \tag{90}$$

and the contravariant basis vectors are

$$e_{\hat{0}} = e^{-\nu}(\partial_t + \omega\partial_\phi) \; , \tag{91}$$
$$e_{\hat{1}} = e^{-\psi}\partial_\phi \; , \tag{92}$$
$$e_{\hat{2}} = e^{-\mu}\partial_R \; , \tag{93}$$
$$e_{\hat{3}} = e^{-\mu}\partial_z . \tag{94}$$

Lecture IV

Exercise 1

An important feature of the Kerr solution is the *frame dragging* which arises ultimately because the metric contains off-diagonal components $g_{t\phi} = g_{\phi t}$. We can compute this effect by considering the trajectory of a particle falling with zero angular momentum, p_ϕ. Since $g_{\alpha\beta}$ is independent of the coordinate ϕ, p_ϕ is still a conserved momenta.

$$p^\phi = g^{\phi\alpha}p_\alpha = g^{\phi\phi}p_\phi + g^{\phi t}p_t \; , \tag{95}$$

and for the time component of the momenta

$$p^t = g^{t\alpha}p_t = g^{tt}p_t + g^{t\phi}p_\phi \; . \tag{96}$$

For a massive particle we have

$$p^t = m\frac{dt}{d\tau} \tag{97}$$
$$p^\phi = m\frac{d\phi}{d\tau} \tag{98}$$

so that the trajectory can be easily derived

$$\frac{d\phi}{dt} = \frac{p^\phi}{p^t} = \frac{g^{\phi t}}{g^{tt}} = \omega(r,\theta) \, , \tag{99}$$

$\omega(r,\theta)$ is the angular velocity of a zero angular momentum particle and it measures the frame dragging. To compute it explicitly we need $g^{\alpha\beta}$, since the metric is diagonal in r and θ

$$g^{rr} = 1/g_{rr} = \frac{\Delta}{\rho^2}\,, \tag{100}$$

$$g^{\theta\theta} = 1/g_{\theta\theta} = \frac{1}{\rho^2}\,, \tag{101}$$

and we are left with the components of the matrix which depends on ϕ and t

$$g^{-1} = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{\phi t} & g_{\phi\phi} \end{pmatrix}^{-1}\,, \tag{102}$$

and given its determinant $D = g_{tt}g_{\phi\phi} - (g_{t\phi})^2$ we are left with the inverse matrix

$$g^{-1} = \frac{1}{D}\begin{pmatrix} g_{tt} & -g_{t\phi} \\ -g_{\phi t} & g_{\phi\phi} \end{pmatrix}\,. \tag{103}$$

With some algebras we obtain

$$D = -\Delta\sin^2\theta\,, \tag{104}$$

$$g^{tt} = \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2\Delta}\,, \tag{105}$$

$$g^{t\phi} = -a\frac{2Mr}{\rho^2\Delta}\,, \tag{106}$$

$$g^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}\,. \tag{107}$$

Inserting this result back into Eq. (99) we obtain

$$\omega(r,\theta) = \frac{g^{\phi t}}{g^{tt}} = \frac{2Mra}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta)}\,. \tag{108}$$

For a stationary observer

$$u^{\mu}_{\rm obs} = \left(u^0_{\rm obs}, 0, 0, 0\right) = \left(\frac{dt}{d\tau}, 0, 0, 0\right)\,, \tag{109}$$

and writing the normalisation $u_{\rm obs}\cdot u_{\rm obs} = -1$,

$$u_{\rm obs}\cdot u_{\rm obs} = g_{00}(u^0_{\rm obs})^2 = -1\,. \tag{110}$$

For the Kerr metric

$$g_{00} = -\left(1 - \frac{2Mr}{\rho^2}\right) = -\left(\frac{r^2 + a^2\cos^2\theta - 2Mr}{r^2 + a^2\cos^2\theta}\right)\,. \tag{111}$$

which is equal to zero if

$$r^2 - 2Mr + a^2\cos^2\theta = 0\,. \tag{112}$$

This quadratic equation for the radius r gives us

$$r_{\rm e}(\theta) = M + \sqrt{M^2 - a^2\cos^2\theta}\,, \tag{113}$$

if $g_{00} = 0$.

If we consider a photon on the equatorial plane ($\theta = \pi/2$) moving a circular orbit ($d\theta = dr = 0$), the line element becomes

$$g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 = 0\,, \tag{114}$$

from which we can compute the angular velocity $\Omega = \frac{d\phi}{dt}$

$$\frac{d\phi}{dt} = \frac{-2g_{t\phi} \pm \sqrt{4g_{t\phi}^2 - 4g_{\phi\phi}g_{tt}}}{2g_{\phi\phi}} \tag{115}$$

$$= -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} \,. \tag{116}$$

By setting $g_{tt} = 0$ we obtain the minimum value of $\Omega_{\rm min} = 0$ while the maximum is

$$\Omega_{\rm max} = -\frac{2g_{t\phi}}{g_{\phi\phi}} > 0 \,. \tag{117}$$

Exercise 2

Let's recall the expression for the orbits of a photon in Schwarzschild

$$\left(\frac{dr}{d\tau}\right) = E^2 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} \,, \tag{118}$$

and the effective potential can be defined as

$$V^2(r) = \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2} \,, \tag{119}$$

and for a static observer The shape of the potential is shown in Fig. (1). For a photon coming from infinity with energy E, the allowed orbits are the ones for which V is smaller than E. At the point where $E^2 = V^2$ we also have

$$\left(\frac{dr}{d\tau}\right)^2 = 0 \,, \tag{120}$$

as indicated by the straight line in Fig. (1). We can evaluate

$$\frac{d}{dr}\left[\left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}\right] = 0 \,, \tag{121}$$

which is satisfied for $r = 3M$ and the potential at this radius is

$$V(r = 3M) = \frac{L}{3\sqrt{3}M} \,. \tag{122}$$

For a photon

$$p^0 = \left(1 - \frac{2M}{r}\right)^{-1} E \,, \tag{123}$$

$$p^r = \frac{dr}{d\lambda} \,, \tag{124}$$

$$p^\phi = \frac{d\phi}{d\lambda} = \frac{L}{r^2} \,. \tag{125}$$

From Eq. (125)

$$\frac{d\phi}{d\lambda} = \frac{d\phi}{d\lambda}\frac{d\lambda}{dr} = \pm\frac{L}{r^2}\frac{1}{\sqrt{E^2 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}}} \tag{126}$$

$$= \pm\frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-1/2} \,, \tag{127}$$

where $b = L/E$ is the *impact parameter*. We compute now the *cone of avoidance* for an incoming photon with $L > 0$. For convenience, we introduce the following variable $u = r^{-1}$ and we rewrite Eq. (127) as the following cubic equation

$$f(u) = \frac{du}{d\phi} = \frac{1}{b^2} - u^2 + 2Mu^3\,. \tag{128}$$

The equation $f(u) = 0$ allows for a negative real root while the two remaining ones may be real or complex and either distinct or coincident (see the lecture notes for the unbound time-like geodesics). The condition for two coincident roots both real can be obtained by

$$f'(u) = 6Mu^2 - 2u = 0\,, \tag{129}$$

which has $u = 1/3M$ as a root; and $u = 1/3M$ is a root of $f(u) = 0$ if

$$b^2 = 27M^2 \quad \text{or} \quad b = (3\sqrt{3})M\,.$$

The product of the roots of Eq. (129) is given by

$$u_1 u_2 u_3 = -\frac{1}{2Mb^2}\,, \tag{130}$$

from which we can finally compute the three roots of the polynomial equation

$$u_1 = -\tfrac{1}{6M} \quad \text{and} \quad u_2 = u_3 = \tfrac{1}{3M}\,.$$

We can use the roots to factorise Eq. (127)

$$\left(\frac{du}{d\phi}\right)^2 = 2M\left(u + \frac{1}{6M}\right)\left(u - \frac{1}{3M}\right)^2\,. \tag{131}$$

Thus if we consider now a bundle of light rays launched from a point at a distance r from the BH, they describe a cone with an angle ψ directed toward the BH

$$\cot\,\psi = \frac{1}{r}\frac{d\tilde{r}}{d\phi}\,, \tag{132}$$

where $d\tilde{r}$ is the line element along the generator of the cone

$$d\tilde{r} = \left(1 - \frac{2M}{r}\right)^{-1/2}\,. \tag{133}$$

If we substitute $u = 1/r$ in Eq. (132)

$$\cot\,\psi = -\frac{1}{u\sqrt{1-2Mu}}\frac{du}{d\phi}\,, \tag{134}$$

and using Eq. (131),

$$\cot\,\psi = -\frac{1}{\sqrt{\frac{r}{2M} - 1}}\left(1 - \frac{r}{3M}\right)\left(1 + \frac{r}{6M}\right)^{1/2}\,. \tag{135}$$

From this last equation it follows that for an ingoing photon

$$\begin{array}{rcl} \psi \sim \frac{3\sqrt{3}}{r}M & \text{as} & r \to \infty\,, \\ \psi = \frac{\pi}{2} & \text{for} & r = 3M\,, \\ \psi = \pi & \text{for} & r = 2M\,. \end{array}$$

It can be easily found similar conditions for an outgoing photon.

Exercise 3

Given a special observer (not moving along a geodesic)

$$u^\mu = \left(u^t, 0, \omega A, 0\right) \,, \tag{136}$$

for which we can compute the energy of a particle measure by this observer as

$$\begin{aligned} E_{\text{ZAMO}} &= -\underline{p} \cdot \underline{u} && (137)\\ &= -\left(p_t u^t + p_\phi u^\phi\right) && (138)\\ &= A(E - wL)\,, && (139) \end{aligned}$$

where A can be found by the normalisation condition $\underline{u} \cdot \underline{u} = -1$

$$A^2 = \frac{g_{\phi\phi}}{-\Delta \sin^2\theta} \,. \tag{140}$$

TODO add the expression for l.

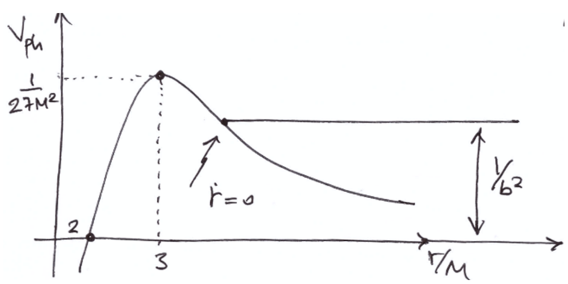

Figure 1: Effective potential for photons in Schwarzschild

Lecture V

Exercise 1

Show that the specific energy and specific angular momentum for circular orbits of massive particles in a Kerr spacetime are given by

$$E = \frac{r^2 - 2\,Mr \pm a\sqrt{Mr}}{r\sqrt{r^2 - 3\,Mr \pm 2\,a\sqrt{Mr}}} \,, \tag{141}$$

$$L = \frac{\sqrt{Mr}\left(r^2 \pm 2\,a\sqrt{Mr} + a^2\right)}{r\sqrt{r^2 - 3\,Mr \pm 2\,a\sqrt{Mr}}} \,. \tag{142}$$

For a circular orbit the potential can be written as

$$-(a^2u^2 - 2Mu + 1) + E^2 + 2Mx^2u^3 - (x^2 + 2aEx)u^2 = 0 \,, \tag{143}$$

where we have defined $x = L - aE$ and $u = 1/r$. The first derivative respect to the radial coordinate, $V'(r) = 0$

$$-(a^2u - M) + 3Mx^2u^2 - (x^2 + 2aEx)u = 0 \,. \tag{144}$$

The cubic polynomial in Eq. (143) has a double root. From Eq. (144) we obtain

$$(x^2 + 2aEx)u = -(a^2u - M) + 3Mx^2u^2 \ , \tag{145}$$

and by solving Eq. (143) for E

$$\begin{aligned} E^2 &= -2Mu + 1 - 2Mx^2u^3 + 3Mx^2u^3 + Mu \\ &= Mx^2u^3 - Mu + 1 \ . \end{aligned} \tag{146}$$

By eliminating E from the previous equations we obtain an equation fro x

$$x^4u^2\left[(3Mu-1)^2 - 4a^2Mu^3\right] - 2x^3u\left[(3Mu-1)(a^2u-M) - 2a^2u(Mu-1)\right] + (a^2u-M)^2 = 0 \ . \tag{147}$$

The discriminant of the last equation is $4a^2M\Delta_\mu^2 u$ where $\Delta_\mu^2 = 1^2u^2 - 2Mu + 1$. We can rewrite Eq. (147) in a particularly simple form by writing

$$(3Mu-1)^2 - 4a^2Mu^3 = Q_+Q_- \ , \tag{148}$$

where $Q_\pm = 1 - 3Mu \pm 2a\sqrt{Mu^3}$. Thus we find

$$x^2u^2 = \frac{Q_\pm\Delta_\mu - Q_+Q_-}{Q_+Q_-} = \frac{1}{Q_\mp}\left(\Delta_\mu - Q_\mp\right) \ . \tag{149}$$

The term in brackets on the RHS of the previous equation can be written

$$\Delta_\mu - Q_\mp = u(a\sqrt{u} \pm \sqrt{M})^2 \ , \tag{150}$$

and the solution of Eq. (147) takes the following from

$$x = -\frac{a\sqrt{u} \pm \sqrt{M}}{\sqrt{uQ_\mp}} \ , \tag{151}$$

where the orbit with the minus sign are retrograde while the one with the plus are direct ones. Inserting x in Eq. (146) we find the expression for the energy

$$E = \frac{1}{\sqrt{Q_\mp}}\left[1 - 2Mu \mp a\sqrt{Mu^3}\right] \ , \tag{152}$$

and easily

$$L = aE + x = \mp\frac{\sqrt{M}}{\sqrt{uQ_\mp}}\left[a^2u^u + 1 \pm 2a\sqrt{Mu^3}\right] \ , \tag{153}$$

as requested.

Exercise 2

We set the energy for a particle at spatial infinity (measured by a static ZAMO with four momentum $u = (1, 0, 0, 0)$) to be equal to unity and calculate the position of the ISCO (depending on a). Bound circular orbits computed in the previous exercises are not all stable. Stability requires that $V''(r) \leq 0$, which yields to the following conditions

$$1 - E^2 \geq \frac{2M}{3r} \ , \tag{154}$$

$$r^2 - 6Mr \pm 8a\sqrt{Mr} - 3a^2 \geq 0 \ , \tag{155}$$

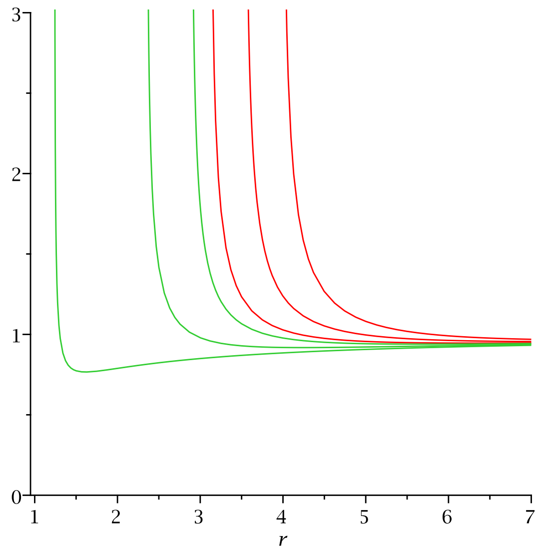

Figure 2: Specific energy for different values of the spin parameter, $a = 0.1, 0.5, 0.98$ and $M = 1$, (in red the corotating orbits and in green counterrotating ones)

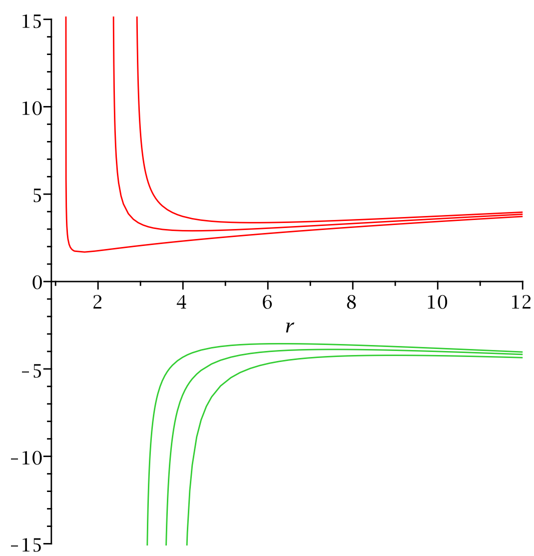

Figure 3: Specific angular momentum for different values of the spin parameter, $a = 0.1, 0.5, 0.98$ and $M = 1$(in red: corotating orbits and in green: counterrotating ones)

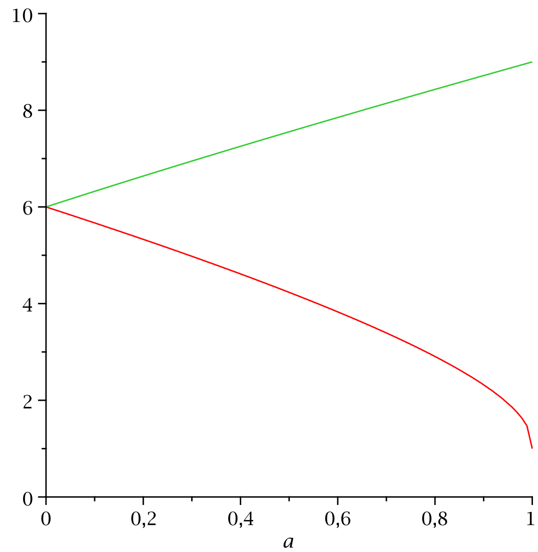

Figure 4: Radius of the ISCO as a function of the spin parameter around a Kerr black hole with $M = 1$(in red: corotating orbits and in green: counterrotating ones)

or equivalently to

$$r \geq R_{\rm ms} \tag{156}$$

where $R_{\rm ms}$ is the radius of the marginally stable orbit, and it is given by the following expression

$$R_{\rm ms} = M \left\{ 3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2} \right\} \tag{157}$$

$$Z_1 \equiv 1 + \left(1 - \frac{a}{M^2}\right)^{1/3} \left[\left(1 + \frac{a}{M}\right)^{1/3} + (1 - \frac{a}{M}\right)^{1/3} \right] \tag{158}$$

$$Z_2 \equiv \left(3 \frac{a^2}{M^2} + Z_1^2\right)^{\mathbf{1}/2} \quad . \tag{159}$$

For $a = 0$, $R_{\rm ms} = 6M$ and for $a = 1$ $R_{\rm ms} = M$ (direct) or $9M$ (retrograde), see Fig. (4). Inserting the expression for the radius of the marginally stable circular orbit in Eq. (152) we obtain a pretty complicated expression for the energy loss (see Matthias' Maple notebook). The smallest energy drop is for a value of the spin parameter $a = 0$, as expected, ($E_{\rm loss} = 0.057$) while the largest one is for $a = 1$ ($E_{\rm loss} = 0.421$).

Exercise 3

Given the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \ , \tag{160}$$

and the Euler-Lagrange equation for r

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \ , \tag{161}$$

we can write it down as

$$\frac{d}{d\lambda} \left(g_{rr} \dot{r}\right) = \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial r} \dot{x}^\mu \dot{x}^\nu \ . \tag{162}$$

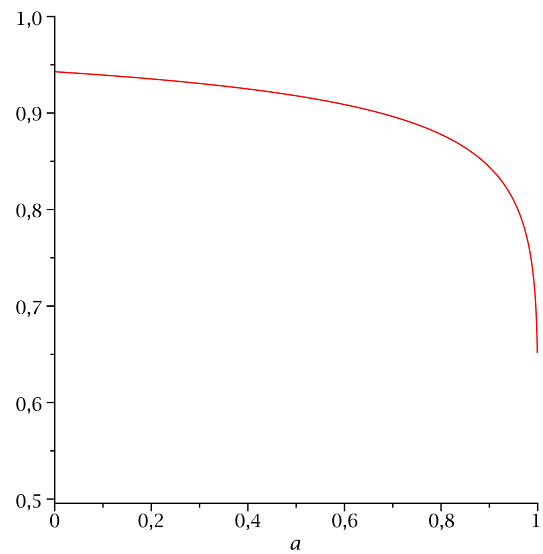

Figure 5: Energy loss for a particle at the ISCO falling into a Kerr black hole with $M = 1$

For a circular orbit we obtain

$$\frac{\partial g_{tt}}{\partial r}\dot{t}^2 + 2\frac{\partial g_{\phi t}}{\partial r}\dot{t}\dot{\phi} + \frac{\partial g_{\phi\phi}}{\partial r}\dot{\phi}^2 = 0 , \tag{163}$$

and given the angular velocity $\omega = \dot{\phi}/\dot{t}$

$$\frac{\partial g_{\phi\phi}}{\partial r}\omega^2 + 2\frac{\partial g_{t\phi}}{\partial r}\omega + \frac{\partial g_{tt}}{\partial r} = 0 . \tag{164}$$

Recalling coefficients of the Kerr metric we obtain

$$\left(r^3 - Ma^2\right)\omega^2 + 2Ma\omega - M = 0 , \tag{165}$$

which has as discriminant

$$M^2a^2 + M(r^3 - Ma^2) = Mr^3 , \tag{166}$$

and the solutions are

$$\omega_\pm = \pm\frac{\sqrt{M}}{r^{3/2} \pm a\sqrt{M}} , \tag{167}$$

which for $a = 0$ the last equation becomes

$$\omega_\pm = \pm\sqrt{\frac{M}{r^3}} . \tag{168}$$

Lecture VI

See Matthias' notes in Maple.

Lecture VIII-IX

Exercise 1

See hand written notes on MS.

Exercise 2

We start by comparing the line elements of the two metrics. We recall that the line element for Misner-Sharp is

$$ds^2 = -a^2(r,t)dt^2 + b^2(r,t)dr^2 + R(r,t)d\Omega^2\,. \tag{169}$$

We choose first a gauge in which $a^2 = 1$, so called *comoving observer gauge*. We assume also that R is separable in a time dependent part and a radial one, $R(r,t) = S(t)\tilde{R}(r)$, which leads to

$$ds^2 = -dt^2 + S^2(t)\left(\frac{b^2}{S^2}dr^2 + \tilde{R}^2 d\Omega^2\right)\,, \tag{170}$$

and by comparing with the FRW metric

$$b(r,t) = \frac{S(t)}{\sqrt{1-kr^2}}\,. \tag{171}$$

The assumption of an homogeneous fluid with non-zero pressure gives us the following conditions to plug into the MS equations

$$D_r p = 0\,, \tag{172}$$

$$p \neq 0\,, \tag{173}$$

$$e(r) = \text{const.} \tag{174}$$

The MS equations become

$$\partial_t U = -\left[\frac{M}{R^2} + 4\pi R\rho\right]\,, \tag{175}$$

$$D_r R = \Gamma\,, \tag{176}$$

where $D_t U = \partial_t U$ give that $a = 1$ and $\Gamma = 1 + U^2 - \frac{2m}{R}$.

Exercise 2

The areal radius of the radially outgoing photons satisfies

$$r_{\rm s} = a\sin\chi = \frac{a_{\rm m}}{2}(1+\cos\eta)\sin[\chi_{\rm e} \pm (\eta - \eta_{\rm e})]\,, \tag{177}$$

The condition $\frac{[\mathcal{A}}{d\eta} \leq 0$ is equivalent to

$$\left.\frac{dr}{d\eta}\right|_{\eta=\eta_e} \leq 0\,. \tag{178}$$

Using the expression for r_s

$$-\sin\eta_e\sin\chi_e + \cos\chi_e + \cos\eta_e\cos\chi_e \leq 0 \iff \tag{179}$$

$$\cos(\eta_e + \chi_e) + \cos\eta_e \leq 0 \iff \tag{180}$$

$$\pi - \chi_e \leq \eta_e + \chi_e\,.,\tag{181}$$

and finally $\eta_e \geq \pi - 2\chi_e$.

Exercise 3

Lecture X

Exercise 1

$$v^i = \frac{\gamma^i_\mu u^\mu}{-n_\mu u^\mu} = \frac{\delta^i_\mu u^\mu + n^i n_\mu u^\mu}{\alpha u^t} \tag{182}$$

$$= \frac{u^i + (-\beta^i/\alpha)(-\alpha)u^t}{\alpha u^t} = \frac{1}{\alpha u^t}\left(u^i + \beta^i u^t\right) \ . \tag{183}$$

Exercise 2

By simply comparing the two metrics

$$\alpha = \frac{1-\frac{M}{2r}}{1+\frac{M}{2r}} \tag{184}$$

and the shift vector is $\beta^i = 0$. The three metric γ

$$\gamma_{ij} = \left(1+\frac{M}{2r}\right)^4 \mathrm{diag}\left(1, r^2, r^2\sin^2\theta\right) \tag{185}$$

and

$$n^\mu = \left(\alpha^{-1}, -\alpha^{-1}\beta^i\right)$$

$$n_\mu = (-\alpha, 0, 0, 0) \ . \tag{186}$$

Exercise 3 (courtesy of Ziri Younsi)

First, consider the metric in 3+1 form

$$\mathrm{ds}^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{187}$$

$$= g_{00}\mathrm{dt}^2 + 2g_{0i}\mathrm{dtd}x^i + g_{ij}\mathrm{d}x^i\mathrm{d}x^j \ . \tag{188}$$

The covariant components of the metric tensor are given by

$$g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{bmatrix}. \tag{189}$$

To complete the derivation we must first derive the relationship between the momentum and the shift. Consider the following

$$\begin{aligned} \frac{\mathrm{d}x_i}{\mathrm{d}\lambda} &= p_i \\ &= g_{i\mu}p^\mu \\ &= g_{i0}p^0 + g_{ij}p^j \\ &= \gamma_{ij}\beta^j p^0 + \gamma_{ij}p^j \ . \end{aligned} \tag{190}$$

Now multiply both sides of equation (4) by γ^{ki}:

$$\gamma^{ki}p_i = \gamma^{ki}\gamma_{ij}\beta^j p^0 + \gamma^{ki}\gamma_{ij}p^j \ .$$

Using the identity $\gamma^{ki}\gamma_{ij} = \delta^k_j$ we obtain

$$\gamma^{ki} p_i = p^k + \beta^k p^0 \ . \tag{191}$$

From equation (5) we obtain the following important identities which will be needed later:

$$p^i = \gamma^{ki} p_k - \beta^i p^0 \ , \tag{192}$$
$$p^j = \gamma^{lj} p_l - \beta^j p^0 \ , \tag{193}$$

which upon rearranging yield the following expressions for the shift

$$\beta^i p^0 = \gamma^{ki} p_k - p^i \ , \tag{194}$$
$$\beta^j p^0 = \gamma^{lj} p_l - p^j \ . \tag{195}$$

Note that equation (7) is identical to equation (6) in Hughes et al. (1994) and may be rewritten as

$$\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} = \gamma^{ij} p_i - \beta^j p^0 \ , \tag{196}$$

where we have let $l \to i$. We are now in a position to derive the expression for p^0. Consider the following

$$\begin{aligned} 0 &= p_\mu p^\mu \\ &= g_{\mu\nu} p^\mu p^\nu \\ &= \left(-\alpha^2 + \beta_i \beta^i\right) \left(p^0\right)^2 + 2\beta_i p^i p^0 + \gamma_{ij} p^i p^j \ . \end{aligned} \tag{197}$$

Bringing $(\alpha p^0)^2$ to the LHS and raising the index on covariant shift terms using γ_{ij} we obtain

$$\begin{aligned} \left(\alpha p^0\right)^2 &= \gamma_{ij} \left[\left(\beta^i p^0\right)\left(\beta^j p^0\right) + 2p^i \left(\beta^j p^0\right) + p^i p^j\right] \\ &= \gamma_{ij} \left[\beta^j p^0 \left(\beta^i p^0 + 2p^i\right) + p^i p^j\right] \ . \end{aligned} \tag{198}$$

Using equations (8)–(9), equation (12) becomes

$$\begin{aligned} \left(\alpha p^0\right)^2 &= \gamma_{ij} \left[\left(\gamma^{lj} p_l - p^j\right)\left(\gamma^{ki} p_k + p^i\right) + p^i p^j\right] \\ &= \gamma_{ij} \left[\gamma^{lj}\gamma^{ki} p_l p_k - \gamma^{ki} p^j p_k + \gamma^{lj} p^i p_l\right] \\ &= \gamma^{lk} p_l p_k - \delta^k_j p^j p_k + \delta^l_i p^i p_l \ . \end{aligned} \tag{199}$$

The second and third terms in equation (13) cancel as k and l are dummy indices and we obtain, upon letting $l \to i$, $k \to j$, the result

$$p^0 = \frac{\sqrt{\gamma^{ij} p_i \, p_j}}{\alpha} \ , \tag{200}$$

which is identical to equation (4) in Hughes et al. (1994), as required. Now consider the Euler-Lagrange equations of motion:

$$\frac{\partial \mathcal{L}}{\partial x^k} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^k}\right) \ , \tag{201}$$

where the Lagrangian is defined as

$$2\mathcal{L} = g_{\mu\nu} p^\mu p^\nu \tag{202}$$
$$= g_{00}\left(p^0\right)^2 + 2g_{0i} p^0 p^i + g_{ij} p^i p^j \ . \tag{203}$$

Using equation (16), the RHS of equation (15) may be calculated as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^k}\right) &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial p^k}\right) \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(2g_{\mu\nu}p^\mu\frac{\partial p^\nu}{\partial p^k}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\mu\nu}p^\mu\delta^\nu_k\right) \\ &= \frac{\mathrm{d}p_k}{\mathrm{d}\lambda} \ ,\end{aligned}$$

hence we obtain the following result:

$$\frac{\partial \mathcal{L}}{\partial x^k} = \frac{\mathrm{d}p_k}{\mathrm{d}\lambda} \ . \tag{204}$$

Finally, we must compute the LHS of equation (15) using the expression for the Lagrangian given in equation (17). We obtain the following:

$$\frac{\partial \mathcal{L}}{\partial x^k} = \frac{1}{2}\left[-2\alpha\alpha_{,k}\left(p^0\right)^2 + \left(\gamma_{ij}\beta^i\beta^j\right)_{,k}\left(p^0\right)^2 + 2\left(\gamma_{ij}\beta^j p^i\right)_{,k}p^0 + \gamma_{ij,k}p^ip^j\right] \ . \tag{205}$$

Now we collect terms in γ_{ij} and $\gamma_{ij,k}$:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x^k} &= -\alpha\alpha_{,k}\left(p^0\right)^2 + \gamma_{ij}\left[\frac{1}{2}\left(\beta^i\beta^j\right)_{,k}\left(p^0\right)^2 + \left(\beta^j p^i\right)_{,k}p^0\right] + \gamma_{ij,k}\left[\frac{1}{2}\beta^i\beta^j\left(p^0\right)^2 + \beta^jp^ip^0 + \frac{1}{2}p^ip^j\right] \\ &= -\alpha\alpha_{,k}\left(p^0\right)^2 + \frac{1}{2}\gamma_{ij}\left[\left(\beta^i\beta^j\right)_{,k}\left(p^0\right)^2 + 2\beta^j_{,k}p^0p^i\right] + \frac{1}{2}\gamma_{ij,k}\left[\beta^i\beta^j\left(p^0\right)^2 + 2\beta^jp^ip^0 + p^ip^j\right] \\ &= -\alpha\alpha_{,k}\left(p^0\right)^2 + T_1 + T_2 \ ,\end{aligned} \tag{206}$$

where

$$T_1 \equiv \frac{1}{2}\gamma_{ij}\left[\left(\beta^i\beta^j\right)_{,k}\left(p^0\right)^2 + 2\beta^j_{,k}p^0p^i\right] \ , \tag{207}$$

$$T_2 \equiv \frac{1}{2}\gamma_{ij,k}\left[\beta^i\beta^j\left(p^0\right)^2 + 2\beta^jp^ip^0 + p^ip^j\right] \ . \tag{208}$$

First let us turn our attention to equation (21). We may expand and subsequently simplify as follows:

$$\begin{aligned}T_1 &= \frac{1}{2}\gamma_{ij}\left[2\beta^i\beta^j_{,k}\left(p^0\right)^2 + 2\beta^j_{,k}p^0p^i\right] \\ &= \gamma_{ij}\left(\beta^ip^0 + p^i\right)\beta^j_{,k}p^0 \\ &= \gamma_{ij}\gamma^{ki}p_k\beta^j_{,k}p^0 \\ &= \beta^j_{,k}p_jp^0 \ ,\end{aligned} \tag{209}$$

as in term 2 on the RHS of equation (5) in Hughes et al. (1994). Secondly, we turn our attention to equation (22). First we may rewrite the expression as follows:

$$T_2 = \frac{1}{2}\gamma_{ij,k}\left[\beta^jp^0\left(\beta^ip^0 + 2p^i\right) + p^ip^j\right] \ . \tag{210}$$

Now we employ equations (8)–(9), yielding:

$$\begin{aligned}T_2 &= \frac{1}{2}\gamma_{ij,k}\left[\left(\gamma^{lj}p_l - p^j\right)\left(\gamma^{mi}p_m + p^i\right) + p^ip^j\right] \\ &= \frac{1}{2}\gamma_{ij,k}\left[\gamma^{mi}\gamma^{lj}p_mp_l + \gamma^{lj}p^ip_l - \gamma^{mi}p^jp_m\right] \ .\end{aligned} \tag{211}$$

At this point we must consider transforming $\gamma_{ij,k}$ into $\gamma^{ij}{}_{,k}$. Consider the following identity:

$$\gamma^{lm} = \gamma^{mi}\gamma^{lj}\gamma_{ij} \ . \tag{212}$$

Differentiating w.r.t. x^k and rearranging yields the following identity:

$$\begin{aligned} \gamma_{ij,k}\left(\gamma^{mi}\gamma^{lj}\right) &= \gamma^{lm}{}_{,k} - \gamma_{ij}\left(\gamma^{mi}\gamma^{lj}\right)_{,k} \\ &= \gamma^{lm}{}_{,k} - \gamma_{ij}\left(\gamma^{lj}{}_{,k}\gamma^{mi} + \gamma^{lj}\gamma^{mi}{}_{,k}\right) \\ &= \gamma^{lm}{}_{,k} - \gamma^{lj}{}_{,k}\delta^m_j - \gamma^{mi}{}_{,k}\delta^l_i \ . \end{aligned} \tag{213}$$

Before proceeding further, notice how terms two and three in equation (25) cancel: symmetry via $\gamma_{ij} = \gamma_{ji}$ and the fact that l and m are dummy indices reveals this. As such we are now left with

$$\begin{aligned} T_2 &= \frac{1}{2}\gamma_{ij,k}\gamma^{mi}\gamma^{lj}p_m p_l \\ &= \frac{1}{2}\left(\gamma^{lm}{}_{,k}\, p_m p_l - \gamma^{lj}{}_{,k}\delta^m_j\, p_m p_l - \gamma^{mi}{}_{,k}\delta^l_i\, p_m p_l\right) \\ &= \frac{1}{2}\left(\gamma^{lm}{}_{,k}\, p_m p_l - \gamma^{lj}{}_{,k}\, p_j p_l - \gamma^{mi}{}_{,k}\, p_m p_i\right) \ . \end{aligned} \tag{214}$$

Note that l, m, i and j are all dummy indices. The choice $j \to m$, $i \to l$ along with symmetry of the metric upon interchange of indices simplifies equation (28), yielding

$$T_2 = -\frac{1}{2}\gamma^{lm}{}_{,k}\, p_m p_l \ . \tag{215}$$

Finally, combining equations (20), (23) and (29), remembering the identity in equation (18), and letting $k \to i$, $j \to k$ we obtain the final result

$$\frac{\mathrm{d}p_i}{\mathrm{d}\lambda} = -\alpha\, \alpha_{,i}\left(p^0\right)^2 + \beta^k_{,i}\, p_k\, p^0 - \frac{1}{2}\gamma^{lm}{}_{,i}\, p_l\, p_m \ , \tag{216}$$

which is precisely the expression in Hughes et al. (1994), as required.

Lecture XI

Exercise 1

$$
\begin{aligned}
\mathcal{L}_{\mathbf{n}}\gamma_{\mu\nu} &= n^\alpha\nabla_\alpha\gamma_{\mu\nu} + \gamma_{\mu\alpha}\nabla_\nu n^\alpha + \gamma_{\nu\alpha}\nabla_\mu n^\alpha \\
&= n^\alpha\nabla_\alpha(n_\mu n_\nu) + g_{\mu\alpha}\nabla_\nu n^\alpha + g_{\nu\alpha}\nabla_\mu n^\alpha \\
&= n^\alpha n_\mu\nabla_\alpha n_\nu + n^\alpha n_\nu\nabla_\alpha n_\mu + \nabla_\nu n_\mu + \nabla_\mu n_\nu \\
&= \gamma^\alpha{}_\mu\nabla_\alpha n_\nu + \gamma^\alpha{}_\nu\nabla_\alpha n_\mu = -2K_{\mu\nu}\,.
\end{aligned}
\tag{217}
$$

Inverting expression (217) and restricting to (nonzero) spatial indices, the third and last expression for the extrinsic curvature is therefore given by

$$
K_{ij} = -\frac{1}{2}\mathcal{L}_{\mathbf{n}}\gamma_{ij}\,. \tag{218}
$$

Exercise 2

I start by providing this ancillary relation that we are going to use in the next exercises. We recall from the lectures that

$$
\alpha := [-(\nabla_\mu t)(\nabla^\mu t]^{-1/2}\,, \tag{219}
$$

$$
n^\mu = -\,\alpha\nabla^\mu t \tag{220}
$$

$$
a_\nu = n^\mu\nabla_\mu n_\nu\,, \tag{221}
$$

if we recall that $\gamma^\beta_\alpha = 0$ we can write

$$
D_\beta\log\alpha = \alpha^{-1}D_\beta\alpha \tag{222}
$$

$$
= \alpha^{-1}\gamma^\gamma_\beta\nabla_\gamma\left\{[-(\nabla_\alpha t)(\nabla^\alpha t)]^{-1/2}\right\} \tag{223}
$$

$$
= \alpha^{-1}\gamma^\gamma_\beta\alpha^3(\nabla_\gamma\nabla_\delta t)(\nabla^\delta t) \tag{224}
$$

$$
= \alpha^2\gamma^\gamma_\beta\left[\nabla_\delta(\alpha^{-1}n_\gamma)\right]n_\delta \tag{225}
$$

$$
= \gamma^\gamma_\beta\nabla_\delta n_c \tag{226}
$$

$$
= a_\beta\,. \tag{227}
$$

Exercise 3

Derivation of the Gauss-Codazzi equation. We introduce an arbitrary vector v^γ tangent to the 3-dimensional hyper surface Σ and we apply the Ricci identity

$$
D_\alpha D_\beta v^\gamma - D_\beta D_\alpha = R^\gamma{}_{\mu\alpha\beta}v^\mu\,. \tag{228}
$$

if we replace D with its definition we obtain

$$
D_\alpha D_\beta v^\gamma = D_\alpha(D_\beta v^\gamma) \tag{229}
$$

$$
= \gamma^\mu_\alpha\gamma^\nu_\beta\gamma^\gamma{}_\rho\nabla_\mu(\gamma^\sigma{}_\nu\gamma^\rho{}_\lambda\nabla_\sigma v^\lambda)\,. \tag{230}
$$

Using the definition of the orthogonal projector $\gamma^\alpha_\beta = \delta^\alpha_\beta + n^\alpha n_\beta$, we write

$$
\nabla_\mu\gamma^\sigma_\nu = \nabla_\mu(\delta^\sigma_\nu + n^\sigma n_\nu) \tag{231}
$$

$$
= \nabla_\mu n^\sigma n_\nu + n^\sigma\nabla_\mu n_\nu\,, \tag{232}
$$

We can replace this expression in Eq. (229) and a bit of algebra we obtain

$$D_\alpha D_\beta v^\gamma = \gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho (n^\sigma \nabla_\mu n_\nu \gamma^\rho_\lambda \nabla_\sigma v^\lambda + \gamma^\sigma_\nu \nabla_\mu n^\rho n_\lambda \nabla_\sigma v^\lambda + \gamma^\sigma_\nu \gamma^\rho_\lambda \nabla_\mu \nabla_\sigma v^\lambda)\,, \tag{233}$$

where the second term inside the parentheses can be replaced by using $\nabla_\sigma(n_\lambda v^\lambda) = 0$. If we introduce the definition of the extrinsic curvature $K_{\beta\alpha} = -\gamma^\mu_\alpha \gamma^\nu_\beta \nabla_\mu n_\nu$

$$D_\alpha D_\beta v^\gamma = -K_{\alpha\beta} \gamma^\gamma_\lambda n^\sigma \nabla_\sigma v^\lambda - K^\gamma_\alpha K_{\beta\lambda} v^\lambda + \gamma^\mu_\alpha \gamma^\sigma_\beta \gamma^\gamma_\lambda \nabla_\mu \nabla_\sigma v^\lambda\,. \tag{234}$$

By considering the symmetry of $K_{\alpha\beta}$, we can write

$$D_\alpha D_\beta v^\gamma - D_\beta D_\alpha v^\gamma = (K_{\alpha\mu} K^\gamma_\beta - K_{\beta\mu} K^\gamma_\alpha) v^\mu + \gamma^\rho_\alpha \gamma^\sigma_\beta \gamma^\gamma_\lambda (\nabla_\rho \nabla_\sigma v^\lambda - \nabla_\sigma \nabla_\rho v^\lambda)\,. \tag{235}$$

The last term corresponds to the Ricci identity for the connection ∇ and therefore we can write

$$D_\alpha D_\beta v^\gamma - D_\beta D_\alpha v^\gamma = \left(K_{\alpha\mu} K^\gamma_\beta - K_{\beta\mu} K^\gamma_\alpha\right) v^\gamma + \gamma^\rho_\alpha \gamma^\sigma_\beta \gamma^\gamma_\lambda R^\lambda_{\ \mu\rho\sigma} v^\mu\ . \tag{236}$$

By substituting the LHS with Eq. (228) we find the requested expression

$$\left(K_{\alpha\mu} K^\gamma_{\ \beta} - K_{\beta\mu} K^\gamma_{\ \alpha}\right) v^\mu + \gamma^\rho_{\ \alpha} \gamma^\sigma_{\ \beta} \gamma^\gamma_{\ \lambda} R^\lambda_{\ \mu\rho\sigma} v^\mu = R^\gamma_{\ \mu\alpha\beta} v^\mu\ , \tag{237}$$

where $\mathbf{v}$ is a vector in the tangent space to the 3-dimensional manifold.

Exercise 4a

Similarly to the previous exercise we apply the Ricci identity although this time we project onto the normal vector $\mathbf{n}$ to the hypersurface, in short $\gamma \cdot \gamma \cdot \gamma \cdot n\ {}^{(4)}R$. We project the Ricci identity onto the 3-dimensional hypersurface

$$\gamma^\mu_{\ \alpha} \gamma^\nu_{\ \beta} \gamma^\gamma_{\ \rho} R^\rho_{\ \sigma\mu\nu} n^\sigma = \gamma^\mu_{\ \alpha} \gamma^\nu_{\ \beta} \gamma^\gamma_{\ \rho} (\nabla_\mu \nabla_\nu n^\rho - \nabla_\nu \nabla_\mu n^\rho)\ . \tag{238}$$

by using the expression derived in the previous exercise, $\nabla_\beta n_\alpha = -K_{\alpha\beta} - a_\alpha n^\beta$, we get

$$\gamma^\mu_{\ \alpha} \gamma^\nu_{\ \beta} \gamma^\gamma_{\ \rho} \nabla_\mu \nabla_\nu n^\rho = \gamma^\mu_{\ \alpha} \gamma^\nu_{\ \beta} \gamma^\gamma_{\ \rho} \nabla_\mu (-K^\rho_{\ \nu} - a^\rho n_\nu) \tag{239}$$

$$= -\gamma^\mu_{\ \alpha} \gamma^\nu_{\ \beta} \gamma^\gamma_{\ \rho} (\nabla_\mu K^\rho_{\ \nu} + \nabla_\mu a^\rho n_\nu + a^\rho \nabla_\mu n_\nu) \tag{240}$$

$$= -D_\alpha K^\gamma_{\ \beta} + a^\gamma K_{\alpha\beta}\ . \tag{241}$$

Therefore, by considering the symmetry properties of $K_{\alpha\beta}$ as in the previous exercise we can easily obtain the Codazzi-Mainardi equations

$$\gamma^\gamma_{\ \rho} n^\sigma \gamma^\mu_\alpha \gamma^\nu_{\ \beta} R^\rho_{\ \sigma\mu\nu} = D_\beta K^\gamma_{\ \alpha} - D_\alpha K^\gamma_{\ \beta}\ . \tag{242}$$

Exercise 4b

The projection onto the hypersurfaces of the Riemann tensor contracted twice with the normal vector results in the Codazzi-Mainardi equations

$$\gamma_\alpha^{\ \epsilon} n^\theta \gamma_\beta^{\ \gamma} n^\eta R_{\epsilon\theta\gamma\eta} = \gamma_\alpha^{\ \epsilon} n^\theta \gamma_\beta^{\ \gamma} \nabla_{[\epsilon} \nabla_{\theta]} n_\gamma \tag{243}$$

$$= -2\gamma_\alpha^{\ \epsilon} n^\theta \gamma_\beta^{\ \gamma} \nabla_{[\epsilon} (K_{\theta]\gamma} + n_{\theta]} D_\gamma \log \alpha) \tag{244}$$

$$= -\gamma_\alpha^{\ \epsilon} n^\theta \gamma_\beta^{\ \gamma} \nabla_\epsilon K_{\theta\gamma} + \gamma_\alpha^{\ \epsilon} n^\theta \gamma_\beta^{\ \gamma} \nabla_\theta K_{\epsilon\gamma} + \gamma_\alpha^{\ \epsilon} \gamma_\beta^{\ \gamma} \nabla_\epsilon D_\gamma \log \alpha \tag{245}$$

$$+ \gamma_\alpha^{\ \epsilon} n^\theta \gamma_\beta^{\ \gamma} (\nabla_\theta n_\epsilon)(D_\gamma \log \alpha) \tag{246}$$

$$= -K^\theta_{\ \alpha} K_{\theta\beta} + n^\theta \nabla_\theta K_{\alpha\beta} - n_\alpha K_{\beta\epsilon} n^\theta \nabla_\theta n^\epsilon - n_\beta K_{\alpha\epsilon} n^\theta \nabla_\theta n^\epsilon \tag{247}$$

$$- n_\alpha n_\beta n^\theta n^\gamma K_{\epsilon\gamma} \nabla_\theta n^\epsilon + \frac{1}{\alpha} D_\alpha D_\beta \alpha\ , \tag{248}$$

where we used the relation, $\nabla_\alpha n_\beta = -K_{\alpha\beta} - n_\alpha D_\beta \log \alpha$. The last step is to rewrite the last expression in terms of the Lie derivative,

$$\mathcal{L}_n K_{\alpha\beta} = n^\gamma \nabla_\gamma K_{\alpha\beta} + K_{\gamma\alpha}\nabla_\beta + K_{\gamma\beta}\nabla_\alpha \, . \tag{249}$$

Using $n^\theta \nabla_\theta n^\epsilon = a^\epsilon$ we can finally obtain

$$\gamma^\epsilon_{\ \alpha} n^\theta \gamma^\gamma_{\ \beta} n^\eta R_{\epsilon\theta\gamma\eta} = \mathcal{L}_n K_{\alpha\beta} + \frac{1}{\alpha} D_\alpha D_\beta \alpha + K_{\alpha\gamma} K^\gamma_{\ \beta} \, , \tag{250}$$

as advertised.

Lecture XII

Exercise 1

We can derive the Hamiltonian constraint starting from the Einstein equations and the Gauss-Codazzi equations derived in the previous section

$$\gamma^{\sigma\gamma}\gamma^\mu_{\ \beta}\gamma^\nu_{\ \delta} R_{\ \sigma\mu\gamma\nu} = R_{\beta\delta} + K K_{\beta\delta} - K_{\beta\gamma} K^\rho_{\ \delta} \, , \tag{251}$$

we contract the indices γ and α

$$\gamma^{\sigma\gamma}\gamma^{\mu\nu} R_{\ \sigma\mu\gamma\nu} = R + K^2 - K_{\alpha\beta}K^{\alpha\beta} \, , \tag{252}$$

Contracting the Codazzi-Mainardi equation gives

$$\gamma^{\sigma\gamma}\gamma^\mu_{\ \beta} n^\delta R_{\ \sigma\mu\gamma\nu} = D_\beta K - D_\alpha K^\alpha_{\ \beta}; , \tag{253}$$

The LHS of Eq. (252) can be expanded

$$\gamma^{\sigma\gamma}\gamma^{\mu\nu} R_{\ \sigma\mu\gamma\nu} = (g^{\sigma\gamma} + n^\sigma n^\gamma)(g^{\mu\nu} + n^\mu n^\nu) R_{\ \sigma\mu\gamma\nu} \tag{254}$$

$$= R + n^\sigma n^\gamma R_{\sigma\gamma} + n^\mu n^\nu R_{\mu\nu} + n^\sigma n^\gamma n^\mu n^\nu R_{\sigma\gamma\nu\mu} \tag{255}$$

$$= R + 2 n^\sigma n^\gamma R_{\sigma\gamma} \, , \tag{256}$$

where we have used the symmetry properties of $R_{\ \sigma\mu\gamma\nu}$. Comparing with the Einstein equations

$$2 n^\sigma n^\gamma G_{\sigma\gamma} = 2 n^\sigma n^\gamma R_{\sigma\gamma} - n^\sigma n^\gamma g_{\sigma\gamma} R \tag{257}$$

$$= 2 n^\sigma n^\gamma R_{\sigma\gamma} - n^\sigma n^\gamma (\gamma_{\sigma\gamma} - n_\sigma n_\gamma) R \tag{258}$$

$$= 2 n^\sigma n^\gamma R_{\sigma\gamma} + R \, . \tag{259}$$

Inserting this equation in Eq. (252) yields

$$2 n^\sigma n^\gamma G_{\sigma\gamma} = R + K^2 - K_{\alpha\beta}K^{\alpha\beta} \, , \tag{260}$$

and if we define the energy density measured by a normal observer, n^α, as $e := n_\alpha n_\beta T^{\alpha\beta}$ we find

$$R + K^2 - K_{\alpha\beta}K^{\alpha\beta} = 16\pi e \, . \tag{261}$$

This is the *Hamiltonian constraint.* Similarly, we can use Eq. (253)

$$\gamma^{\sigma\gamma}\gamma^\mu_{\ \beta} n^\delta R_{\ \sigma\mu\gamma\nu} = \gamma^\mu_{\ \beta} n^\delta R_{\mu\delta} + \gamma^\mu_{\ \beta} n^\sigma n^\gamma n^\delta R_{\ \sigma\mu\gamma\nu} \, . \tag{262}$$

The second term is again zero while the first term on the RHS as

$$\gamma^\mu_{\ \beta} n^\delta G_{\mu\delta} = \gamma^\mu_{\ \beta} n^\delta R_{\mu\delta} - \frac{1}{2}\gamma^\mu_{\ \beta} n^\delta g_{\mu\delta} R \, . \tag{263}$$

Since $\gamma^\mu_{\ \beta} n^\delta g_{\mu\delta} = \gamma_{\beta\delta} n^\delta = 0$, Eq. (253) reduces to

$$\gamma^\mu_{\ \beta} n^\delta G_{\mu\delta} = D_\beta K - D_\alpha K^\alpha_\beta \, . \tag{264}$$

The momentum density can be defined as $j_\alpha := -\gamma^\beta_{\ \alpha} n^\rho T_{\beta\rho}$, which leads finally to the *momentum constraint*

$$D_\beta K^\beta_{\ \alpha} - D_\alpha K = 8\pi j_\alpha \, . \tag{265}$$

Exercise 2

Given the scalar field ϕ

$$\tilde{D}_i\left(\tilde{D}_j\phi\right) = \tilde{D}_i\left(\partial_j\phi\right) = \partial_i\partial_j\phi - \tilde{\Gamma}^k_{\ ij}\partial_k\phi\ , \tag{266}$$

as advertised.

$$\tilde{D}_i\tilde{D}_j\phi = D_iD_j\phi + \frac{1}{\phi}\gamma_{ij}\partial^k\phi\ \partial_k\phi - \frac{2}{\phi}\partial_i\phi\ \partial_j\phi\ . \tag{267}$$

Using the definition of covariant derivative

$$\partial_i\partial_j\phi - \tilde{\Gamma}^k_{\ ij}\ \partial_k\phi = \partial_i\partial_j\phi - \Gamma^k_{\ ij} + \frac{1}{\phi}\gamma_{ij}\partial^k\phi\ \partial_k\phi - \frac{2}{\phi}\partial_i\phi\ \partial_j\phi\ . \tag{268}$$

and rearranging the previous equation

$$\left(\tilde{\Gamma}^k_{\ ij} - \Gamma^k_{\ ij}\right)\partial_k\phi = \frac{2}{\phi}\partial_i\phi\ \partial_j\phi - \frac{1}{\phi}\gamma_{ij}\partial^k\ \phi\partial_k\phi\ . \tag{269}$$

Now we compute explicitly the affine connection coefficients for the conformal metric

$$\tilde{\Gamma}^k_{\ ij} = \frac{1}{2}\tilde{\gamma}^{kl}\left(\partial_i\tilde{\gamma}_{jl} + \partial_j\tilde{\gamma}_{li} - \partial_l\tilde{\gamma}_{ij}\right) \tag{270}$$

$$= \frac{1}{2\phi^2}\gamma^{kl}\left(\partial_i\phi^2\ \gamma_{jl} + \partial_j\phi^2\ \gamma_{li} - \partial_l\phi^2\ \gamma_{ij}\right) \tag{271}$$

$$= \frac{1}{2\phi^2}\gamma^{kl}\left[\phi^2\left(\partial_i\gamma_{jl} + \partial_j\gamma_{li} - \partial_l\gamma_{ij}\right) + \right. \tag{272}$$

$$\left. 2\phi\gamma_{il}\ \partial_i\phi + 2\phi\ \gamma_{li}\partial_j\phi - 2\phi\ \gamma_{ij}\partial_l\phi\ \right] \tag{273}$$

$$= \Gamma^k_{ij} + \frac{1}{\phi}\gamma^{kl}\left(\gamma_{jl}\partial_i\phi + \gamma_{li}\partial_j\phi - \gamma_{ij}\partial_l\phi\right) \tag{274}$$

$$= \Gamma^k_{ij} + \delta^k_j\partial_i\ln\phi + \delta^k_i\partial_j\ln\phi - \gamma_{ij}\gamma^{kl}\partial_l\ln\phi\ , \tag{275}$$

note that this expression differs from Eq. (7.100) in Rezzolla & Zanotti's book. We can substitute this expression in the LHS of Eq. (269) which leads to

$$\frac{1}{\phi}\left(\delta^k_j\partial_i\phi + \delta^k_i\partial_j\phi - \gamma_{ij}\gamma^{kl}\partial_l\phi\right)\partial_k\phi = \tag{276}$$

$$\frac{2}{\phi}\partial_i\phi\ \partial_j\phi - \frac{1}{\phi}\gamma_{ij}\partial^k\phi\ \partial_k\phi\ , \tag{277}$$

which is exactly the RHS of Eq. (269) as requested.

Exercise 3

We define a normal observer $u^\mu = W(1, v^i) = W(n^\mu + v^\mu)$, we can rewrite the stress energy tensor as

$$T^{\mu\nu} = h\rho W^2(n^\mu + v^\mu)(n^\nu + v^\nu) + p(\gamma^{\mu\nu} - n^\mu n^\nu) \tag{278}$$

$$= h\rho W^2(n^\mu n^\nu + v^\mu n^\nu + n^\mu v^\nu + v^\mu v^n u) + p(\gamma^{\mu\nu} - n^\mu n^\nu) \tag{279}$$

$$= \left(h\rho W^2 - p\right)n^\mu n^\nu + h\rho W^2(v^\mu n^\nu + n^\mu v^\nu) + h\rho W^2 v^\mu v^\nu + p\gamma^{\mu\nu} \tag{280}$$

$$= En^{\mu\nu} + h\rho W^2 v^\mu n^\nu + h\rho W^2 v^\nu n^\mu + S^{\mu\nu} \tag{281}$$

$$= En^{\mu\nu} + S^\mu n^\nu + S^\nu n^\mu + S^{\mu\nu}\ , \tag{282}$$

where we have used the definition of the total energy for the Eulerian observe,
the momentum density and the purely spatial energy-momentum tensor.