GENERAL RELATIVITY

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- The goal of this course is to provide an introduction to Einstein's theory of general relativity and to explore the consequences of the best theory of gravity known.

- The course is structured into three main parts
  (1) differential geometry, tensor algebra and calculus
  (2) derivation of Einstein equations
  (3) some examples of exact solutions (bhs, cosmology)

- Special relativity will be introduced as well but will be seen mostly as a special case of the theory.

- Exercise are necessary for the credits and indispensable to understand the theory.

- A course on "Advanced General Relativity" may be offered if there is an interest and time availability

Bibliography

- A first course in GR, Schutz, Cambridge Univ Press, 1993
- Introducing Einstein's relativity, R. d'Inverno Oxford Univ Press 1992
- Gravity, J. Hartle, Addison-Wesley, 2003
- Relativistic Hydrodynamics, LR, O. Zanotti, Oxford Univ Press, 2013 (chapter 1)

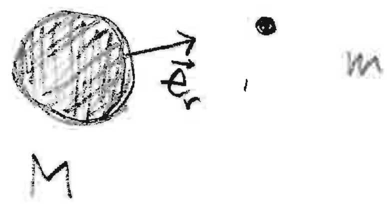
- Handwritten notes will be uploaded after every lecture (see astro.uni-frankfurt/rezzolla/teaching)
- Exercises will be assigned after every lecture (see website)
- Solutions to exercises will be discussed and eventually uploaded (you are encouraged to do the exercises!)
- Do not hesitate to ask! No question is stupid
- You can take notes but make sure you understand what I'm saying.

Why general relativity?

Why do we need a new theory of gravity? What is wrong with the Newtonian theory of gravity? After all, its using this theory that we have buit the houses that we have slept in, the cars that have brought us here, etc.

Let's recall what Newton said. Take two masses M, m seperated by a distance $|\vec{r}|$, there will be a gravitational force between them $\vec{F}_g$

$$\vec{F}_g = \frac{d\vec{p}}{dt} = -\frac{GMm}{r^2}\vec{e}_r$$



$$= -m\vec{\nabla}\phi_g$$

where $\phi_g := -\frac{GM}{r}$ gravitational potential

What's wrong with this picture ?

Well, first of all it doesn't work, ie it cannot be used to explain simply astronomical observations such as the perihelion precession of Mercury, or to obtain a Global Positioning System (GPS).

More seriously, however, provides a desciption of gravity in terms of an "instantaneous" force. This is in stark contrast with all we know in physics, ie that fields propagate at finite velocities and that these velocities are upper limited by the speed of light.

Breaking up with this picture lead Einstein, almost 100 years ago, to the revision of the understanding of gravity and to the Einstein equations

Einstein equations are :

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta} \qquad (1)$$

Understanding their derivation and their implications will require your attention for the rest of the semester, but will also represent an important milestone of a journey we will start today.

Equations (1) are covariant tensor equations ie, they are equations involving tensors and that are invariant under coordinate transformations (covariant).

We need to understand precisely what this means, and we will start from discussing the concept of coordinates

COORDINATES

I'm sure you are all familiar with standard coordinates such as:

- Cartesian (x, y, z)
- Spherical (r, θ, ϕ)
- Cylindrical (ρ, θ, z)
- bispherical (σ, τ, ϕ)
- stereographic $(x, y, \ )$

The motto in general relativity is that "coordinates are only coordinates".

In other words, coordinates are just specific choices that do not carry any physical significance.

At the same time, the choice of "good" coordinates is often essential to carry out calculations, avoid pitfalls (coordinate singularities), obtain numerical solutions.

We will see later how to define coordinates, recognize possible pathologies and cure them. For the time being, however we need to introduce the concept of <u>spacetime</u> as a <u>manifold</u>.

You are probably familiar from classical mechanics with the idea of a <u>global</u> <u>space</u> and a <u>global time</u> as two entities against which the laws of physics (eg conservation of energy and momentum) are defined.

This concept of space and time as a fixed background needs to be abandoned in general relativity. Space and time are simply tools to specify the relative positions of <u>events</u>

What is an <u>"event"</u>? Mathematically, this is an element of the differentiable manifold that we will call <u>spacetime</u>.

Physically, you can think of an event as "something happening somewhere at some time".

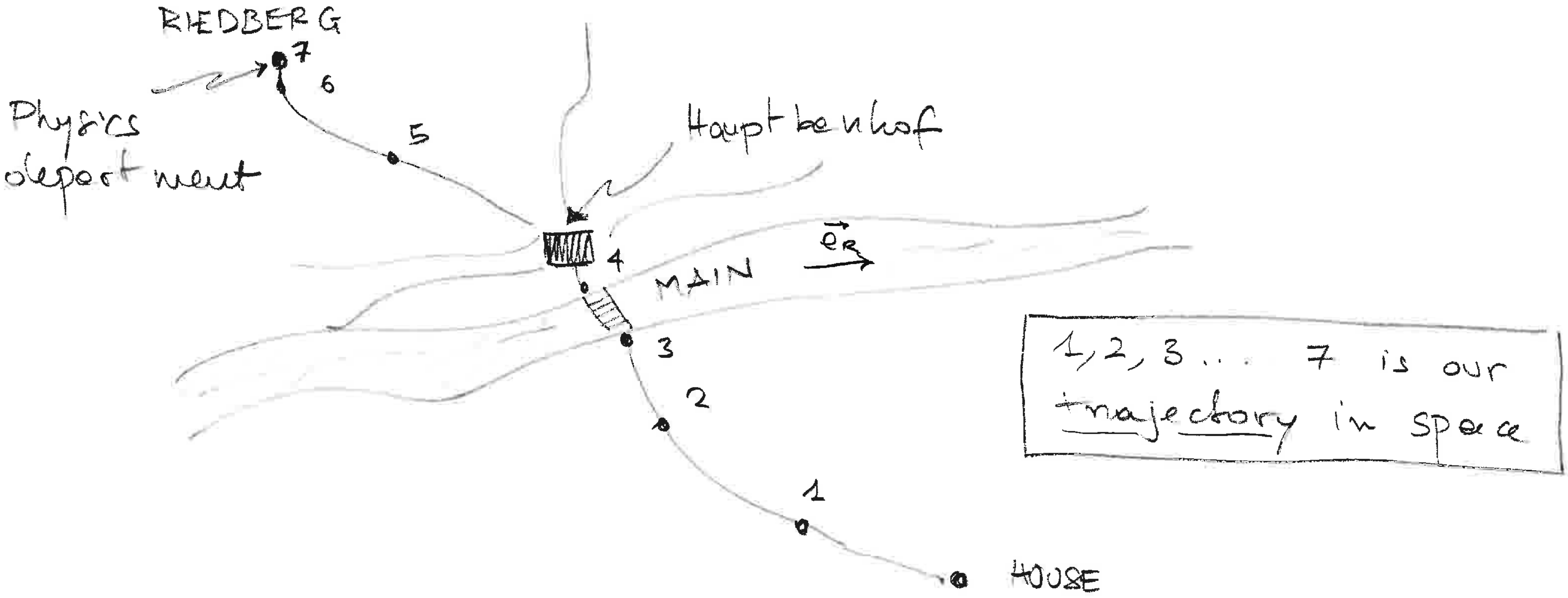
An event is the instant at which a photon from the blackboard is reaching your retina. This is a precise physical process which is however related to other processes (eg, me writing something on the blackboard) and among which we need to find a relation (ie assign coordinates)

We are normally not accustomed to the idea that events should be considered as a sequence of points in a 4D spacetime because our experience tends to create a marked distinction between space and time. We need to break this distinction.

I will now discuss how to extend our discussion of space and time into a description of spacetime ⊙ with a simple example.

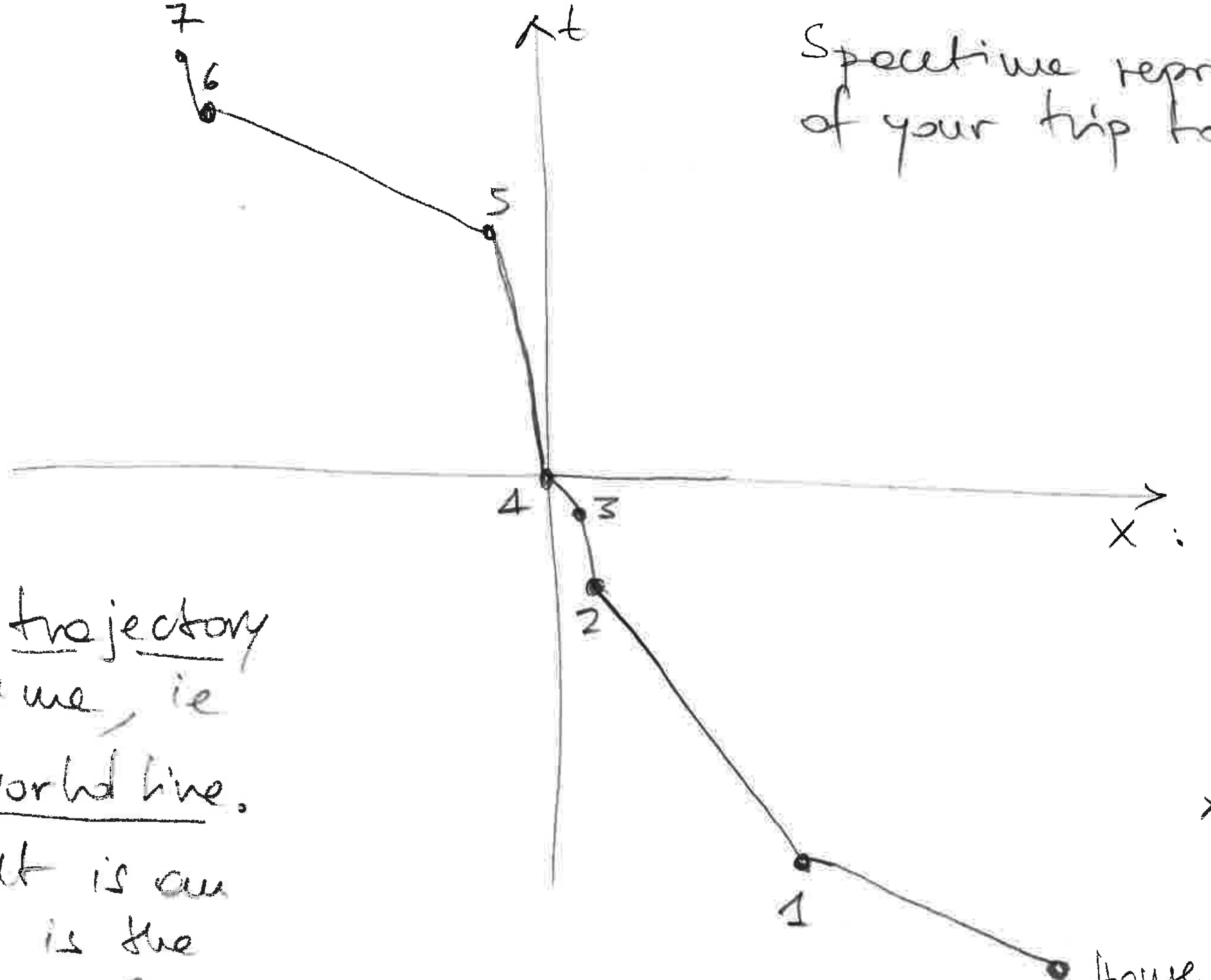
Consider we should describe the trajectory that a student here now has made to go from his house this morning and be here for this lecture.

⊙ writing spacetime and not space-time is useful to consider this as a single object



We have no problem understanding this picture. Indeed we are all super-experts of Google Maps!
However, this is not a helpful way of representing our trip as a sequence of events. Rather, we should think of a spacetime description

Spacetime representation of your trip to the classroom



This is a <u>trajectory</u> in spacetime, ie its a <u>world line</u>. Each point is an event, ie is the combination of space and time.

Note that the segments joining 1, 2, ... 7 are of different lenght and slope. What is the physical significance of the slope of the world line? It's a measure of the velocity

Note that we have made a choice for the time coordinate and the space coordinates (eg distance from Hbf) However these choices are arbitrary and do not play any role. We could have decided to measure things from a coordinate system centered in the Galaxy. This would have been silly but legitimate.

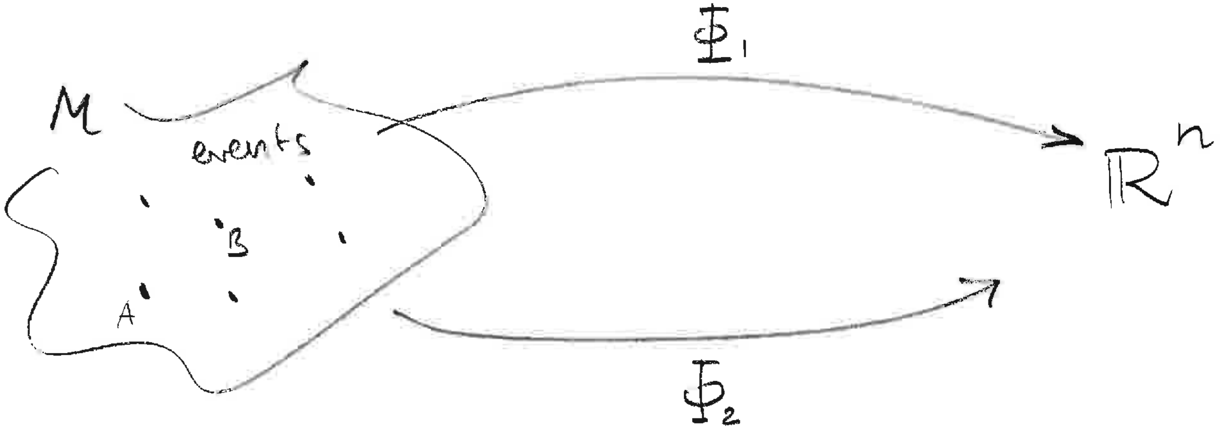
What is important in the spacetime representation is not the exact position among events 1, 2, .. 7. What is important is their distance, $\sqrt{ds^2}$, which we will next learn how to measure.

Hereafter we will treat the spacetime as a differentiable manifold $\mathcal{M}$ of dimension four. In this way we specify the nature of the spacetime as a topological space of Hausdorff type (ie different events in the spacetime allow for disjoint local neighborhoods) and that the parameterisation that we use to 'order' the events in the spacetime are functions of class $\mathcal{C}^{\ell}$ $\ell \geqslant 2$ (ie with second or higher-order derivatives that are continuous).

Stated differently, a differentiable manifold $\mathcal{M}$ of dimension four is a Hausdorff topological space diffeomorphic to $\mathbb{R}^4$.

Since the physical events are the elements of the manifold, we need to make choices for the parameters needed to describe these events. The number of independent

peremeters is represented by the dimension of M.
These peremeters are then the <u>coordinates</u> needed to cover the manifold.



Φ_1, Φ_2 : maps of M to $\mathbb{R}^n$

$$A \longrightarrow (x_A^1, x_A^2, x_A^3, \cdots x_A^n) \quad \text{Coords. of event } A$$
$$B \longrightarrow (x_B^1, x_B^2, \cdots\cdots x_B^n) \quad \text{" \quad " \quad " } B$$

Don't be fooled in thinking that this association of events to coordinates is Euclidean <u>globally</u>. It can be only in very special cases.[1] However, it can always be Euclidean on a small patch, ie locally.

[1] flat spacetime

An example is a torus, which has a non trivial topology but that can always be mapped to an Euclidean space locally (ie the tangent plane to a given point).

Note also that there are infinite many possible mappings (this the gauge freedom in essence) and that there are "good" maps and "bad" maps. Using the good maps is of course essential in many cases

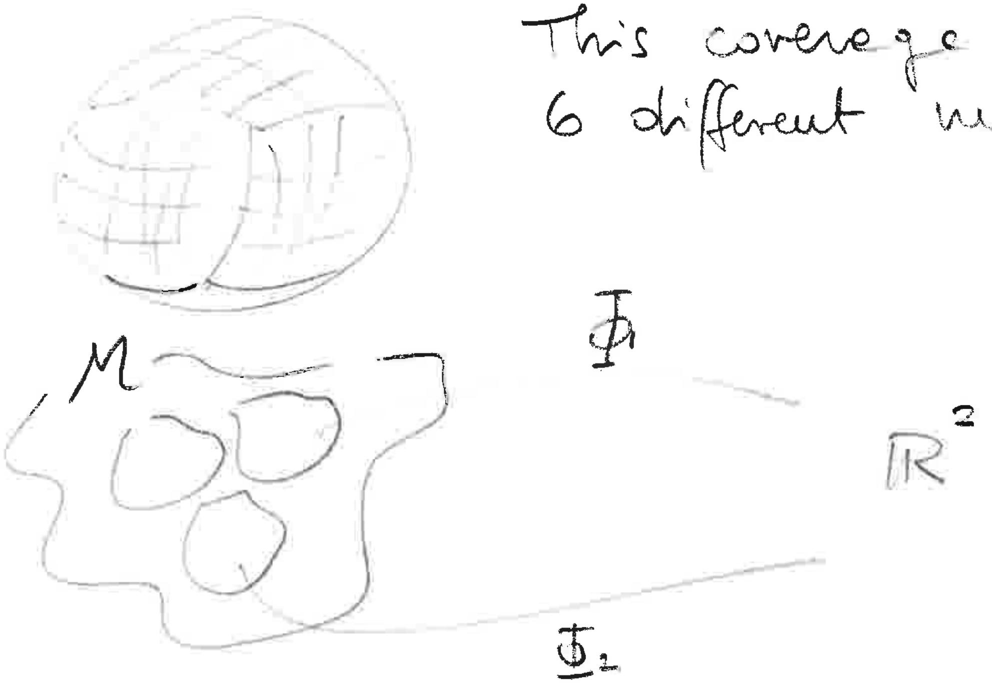
Ex. 2-sphere, ie surface of a sphere in three-space

This surface is normally mapped with two coordinates, eg (θ, ϕ), (latitude, longitude)

Yet this coordinate map is <u>pathological</u> at the poles.

Poles $\theta = 0, \pi$; $\phi \in [0, 2\pi]$ infinite different points are concentrated into one

Better coordinate systems can be used, eg stereographic coordinates



This coverage of the 2-sphere requires 6 different maps or coordinate patches

An "atlas" is then any collection of patches that covers the whole manifold

Note that differentiability of the manifold is an essential property in general relativity

A 2-sphere is a differentiable manifold. A cone is <u>not</u>; there exist a point (the vertex) which is not differentiable.

Recap: we have seen that the spacetime can be seen as a collection of events (differentiable manifold) and that there are infinitely different ways of representing such events. After all coordinates are only coordinates.

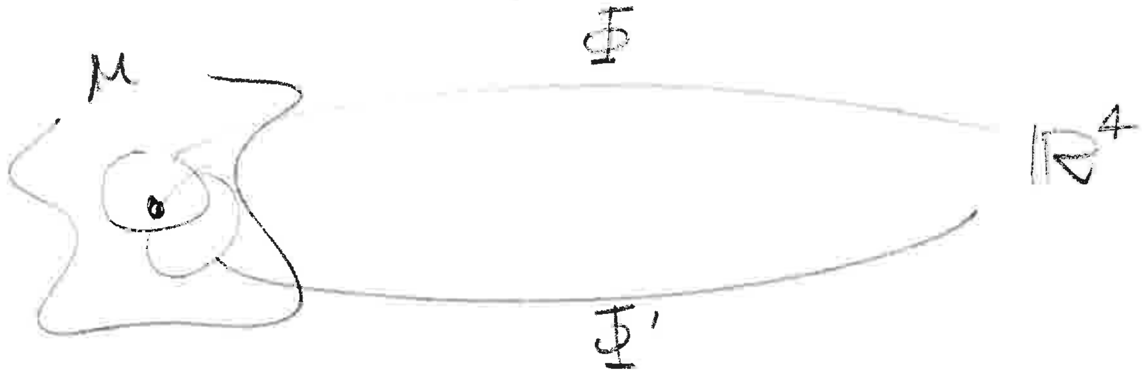
How can we express the law of physics in a way that does not vary when going from one coordinate system to the next?

The property we require is that of <u>covariance</u>, ie the property for which the equations have always the same form independently of the coordinate system chosen.

To obtain covariant equations (ie equations that do not change form) we need to learn how the basic element of a manifold, eg curves, change upon a coordinate transformation.

Hereafter we will specify to a four-dimensional manifold (eg, one with 3 spatial dimensions and 1 temporal dimension).

Consider next a poin P in $\mathcal{M}$ and two different maps Φ and Φ' of P to $\mathbb{R}^4$



Φ will describe P with 4 numbers (coordinates)

$$X^\mu_P = (X^0, X^1, x^2, x^3.)$$

Φ' will describe it with 4 different numbers

$$X^{\mu'}_P = (X^{0'}, X^{1'}, X^{2'}, X^{3'})$$

A <u>coordinate transformation</u> $\{x^\mu\} \longrightarrow \{x^{\mu'}\}$ at P is expressed in terms of 4 functions $f^{\mu'}(x^1, x^2, x^3, x^4)$, ie

$$x^{\mu'} = f^{\mu'}(\underline{x})\Big|_P = f^{\mu'}(x^1, x^2, x^3, x^4)\Big|_P$$

this is a 4-dimensional object in M

Because M is differentiable, the coordinate transformation doesn't need to be restricted to P and hence we can write

$$x^{\mu'} = f^{\mu'}(\underline{x}) \iff \underline{x}' = \underline{f}(\underline{x})$$

The transformation $\underline{f}$ should be invertible, ie such that $\underline{f}^{-1}$ exists and

$$x^\mu = (f^{-1})^\mu (x^{\mu'}) \iff \underline{x} = \underline{f}^{-1}(\underline{x}')$$

of course $f \circ f^{-1} = 1$, ie the combined operation is an identity

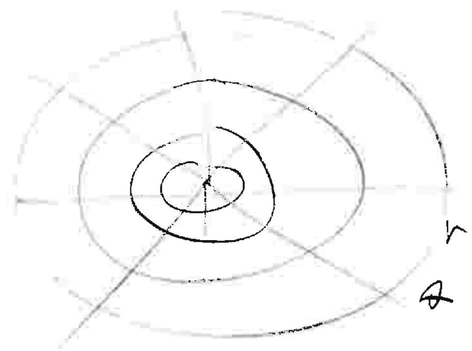
$$x^\mu = x^\mu(x^{\mu'}) = x^\mu(x^{\mu'}(x^\mu))$$

Example

consider a 2-dimensional plane covered with coordinates

$\{x^\mu\} = (x, y)$ and $\{x^{\mu'}\} = (r, \theta)$

x^μ



$x^{\mu'}$

Then the coordinate transformation and its inverse are

$$\underline{f}: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \underline{f}^{-1}: \begin{cases} r = (x^2+y^2)^{1/2} \\ \theta = tg^{-1}\left(\frac{y}{x}\right) \end{cases}$$

of course $f \circ f^{-1} = 1$

$$x = (x^2+y^2)^{1/2} \cos\left(tg^{-1}\left(\frac{y}{x}\right)\right) = x$$

More in general, we can express the changes of the coordinates $\underline{x}'$ with respect to the coordinates $\underline{x}$ in terms of a transformation matrix $\underline{\Lambda}$

$$\Lambda^{\mu'}{}_{\mu} := \frac{\partial x^{\mu'}}{\partial x^{\mu}}$$

in matrix form

$$\Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} \frac{\partial x^{0'}}{\partial x^0} & \frac{\partial x^{0'}}{\partial x^1} & \frac{\partial x^{0'}}{\partial x^2} & \frac{\partial x^{0'}}{\partial x^3} \\ \frac{\partial x^{1'}}{\partial x^0} & \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} & \frac{\partial x^{1'}}{\partial x^3} \\ \frac{\partial x^{2'}}{\partial x^0} & \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} & \frac{\partial x^{2'}}{\partial x^3} \\ \frac{\partial x^{3'}}{\partial x^0} & \frac{\partial x^{3'}}{\partial x^1} & \frac{\partial x^{3'}}{\partial x^2} & \frac{\partial x^{3'}}{\partial x^3} \end{pmatrix}$$

let J' be the determinant of $\Lambda^{\mu'}{}_{\mu}$, ie

$$J' := \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right|$$

If $J' \neq 0$ everywhere, we can obtain the inverse transformation f^{-1}

If $J' = 0$ somewhere, the transformation is said to be <u>singular</u> there

Recap

- General relativity requires the introduction of "spacetime".
- Mathematically this is a differentiable manifold, of dimension 4, ie a Haussdorff topological space diffeomorphic to $\mathbb{R}^4$
- Physically, spacetime is a container of "events", where an event is "something-happening-sometime-somewhere"
- M can be covered by coordinates, associating 4 numbers to each event.
- There are infinitely many different mappings possible: coordinates are not important as physical processes are independent of the coordinates ("coords. are just coords.")

- At the same time, the use of suitable coordinate systems is often essential to distinguish physical and non physical pathologies (eg coordinate singularities or degeneracies)

- End result of GR is the "Einstein equations": covariant tensor equations, ie, equations that do not change form after a coordinate transformation.

- To obtain covariant equations we need to learn how elements in M change under coordinate transformations.

- A coordinate transformation at an event P

$$x^{\mu}(P) \xrightarrow{f} x^{\mu'}(P) \quad : \text{direct}$$

$$x^{\mu'}(P) \xrightarrow{f^{-1}} x^{\mu}(P) \quad : \text{inverse transformation}$$

- $f \circ f^{-1} = \mathbb{1}$

Ex

$\{x^\mu\} = (x, y)$; $\{x^{\mu'}\} = (r, \theta)$

$f: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$; $f^{-1}: \begin{cases} r = (x^2+y^2)^{1/2} \\ \theta = tg^{-1}\left(\frac{y}{x}\right) \end{cases}$

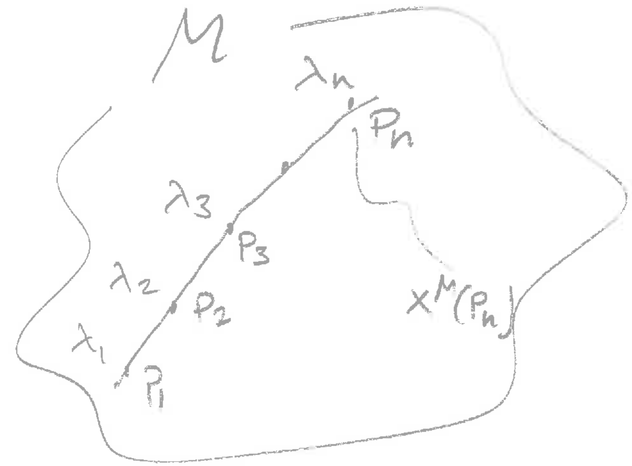
$$\Lambda^{\mu'}{}_{\mu} := \frac{\partial x^{\mu'}}{\partial x^{\mu}} \quad ; \quad \Lambda^{\mu}{}_{\mu'} := \frac{\partial x^{\mu}}{\partial x^{\mu'}}$$: transformation matrices

- We will next learn how to transform the various elements of the manifold $\mathcal{M}$: curves, vectors, function, tensors.

Curves and tangent vectors

We have seen that the events represent the basic components of a manifold. A sequence of nearby events can be collected into a path to be meant as an intrinsic geometrical object, independent of its coordinate representation. The events along a path can then be ordered via a parameter, λ; varying λ in a given interval $I = [a,b] \cap \mathbb{R}$ will select a series of coordinates

$x^\mu(\lambda)$ describing a curve $\mathcal{C}(\lambda)$



$$\mathcal{C}(\lambda) : \{ x^\mu(\lambda), \text{ with } \lambda \in I \cap \mathbb{R} \}$$

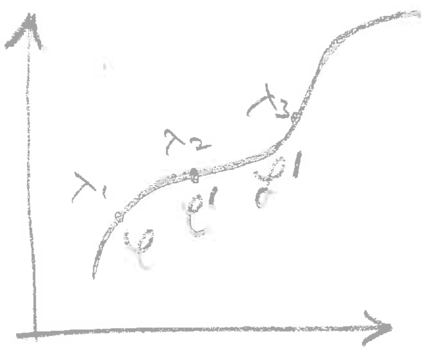
Note that $\mathcal{C}$ will depend both on the choice made for λ and on the coordinates chosen to cover the manifold.

A change of λ will lead to a new "image" of the curve, that is, a new curve passing through the same path and having the same coordinate representation.
Conversely, if a different coordinate system is used, this will lead to a new curve with a different representation.

$$\mathcal{C} : \{x^\mu(\lambda), \lambda \in I \cap \mathbb{R}\} \text{ : curve}$$

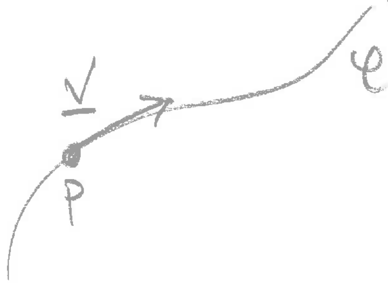
$$\mathcal{C}' : \{x^\mu(\lambda'), \lambda' \in I' \cap \mathbb{R}\} \text{ : new image}$$

$$\mathcal{L}' : \{x^{\mu'}(\lambda'), \lambda' \in I \cap \mathbb{R}\} \text{ : new curve}$$



Having defined a curve, we can next discuss the concept of a <u>tangent vector</u>, which is defined as a mesure of how the coordinates alon $\mathcal{C}(\lambda)$ vary when λ is varied, i.e,

$$V^\mu := \frac{dx^\mu}{d\lambda}\bigg|_P$$

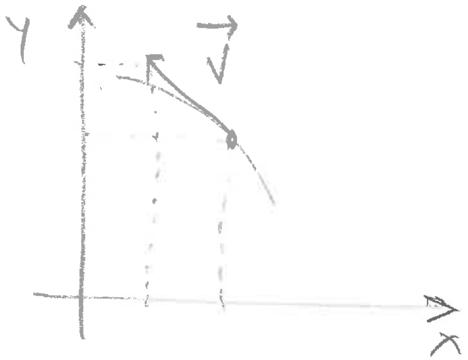


tangent vector to $\mathcal{C}$ at the event P

We can extend this concept to any point along $\mathcal{C}$ and define the <u>generic tangent vector</u> to $\mathcal{C}(\lambda)$ as

$$V^\mu := \frac{dx^\mu}{d\lambda}$$

In analogy to the distinction between path and curve we should distinguish between the tangent vector $\underline{V}$ as a geometrical object and its coordinate representation, which will obviously depend on the choice made for the coordinate map



$V^i = (V^x, V^y)$ in a Cartesian coordinate system (x, y)

$V^i = (0, V^\theta)$ in a polar coordinate system (r, θ)

It's easy now to calculate how the components of $\underline{V}$ vary upon a coordinate transformation $\{x^\mu\} \to \{x^{\mu'}\}$

$$V^{\mu'} = \frac{dx^{\mu'}}{d\lambda} = \sum_{\mu=0}^{3} \frac{\partial x^{\mu'}}{\partial x^\mu} \underbrace{\frac{dx^\mu}{d\lambda}}_{V^\mu} = \sum_{\mu=0}^{3} \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$$

$$dx^{\mu'} = \sum_{\mu=0}^{3} \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$$

Let's write this explicitly for one component, eg, $\mu=1$

$$V^{1'} = \sum_{\mu=0}^{3} \frac{\partial x^{1'}}{\partial x^\mu} V^\mu = \frac{\partial x^{1'}}{\partial x^0} V^0 + \frac{\partial x^{1'}}{\partial x^1} V^1 + \frac{\partial x^{1'}}{\partial x^2} V^2 + \frac{\partial x^{1'}}{\partial x^3} V^3$$

This is simple but can be written in a more compact form; we have seen that $\Lambda^{\mu'}{}_\mu = \frac{\partial x^{\mu'}}{\partial x^\mu}$ so that

$$V^{\mu'} = \sum_{\mu=0}^{3} \Lambda^{\mu'}{}_\mu V^\mu = \Lambda^{\mu'}{}_\mu V^\mu$$

where we have used the convention for <u>summation over repeated indices</u>. This is an important (tensor) equation we should discuss briefly

free contravariant indices

$$\boxed{V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu} \qquad (2)$$

repeated (contracted or dummy) covariant index

Eq (2) is the transformation rule of a <u>contravariant vector</u> V^μ.

We can extend (2) to multiple coordinate transformations

$$\{x^\mu\} \xrightarrow{f} \{x^{\mu'}\} \xrightarrow{g} \{x^{\mu''}\}$$

free index

then

$$V^{\mu''} = \sum_{\mu'=0}^{3} \Lambda^{\mu''}{}_{\mu'} V^{\mu'} = \sum_{\mu'=0}^{3} \sum_{\mu=0}^{3} \Lambda^{\mu''}{}_{\mu'} \Lambda^{\mu'}{}_{\mu} V^\mu$$

dummy indices

$$= \Lambda^{\mu''}{}_{\mu'} \Lambda^{\mu'}{}_{\mu} V^\mu$$

μ'': only free index

μ, μ' are dummy indices

$$= \Lambda^{\mu''}{}_{\alpha'} \Lambda^{\alpha'}{}_{\beta} V^\beta$$

note how much simpler the last expression is

Of course, we can also consider the inverse transformation

$$\{x^{\mu'}\} \longrightarrow \{x^{\mu}\}$$

$$V^{\mu} = \Lambda^{\mu}{}_{\mu'} V^{\mu'}$$

where the matrix $\Lambda^{\mu}{}_{\mu'}$ is the inverse of $\Lambda^{\mu'}{}_{\mu}$, ie

$$\Lambda^{\mu'}{}_{\mu} \Lambda^{\mu}{}_{\nu'} = \delta^{\mu'}{}_{\nu'} = \begin{cases} 0 & \text{if } \nu' \neq \mu' \\ 1 & \text{if } \nu' = \mu' \end{cases}$$ Kronecker delta

and

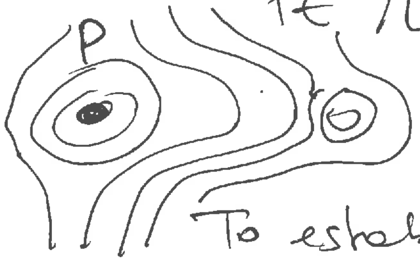
$$\Lambda^{\mu}{}_{\mu'} \Lambda^{\mu'}{}_{\nu} = \delta^{\mu}{}_{\nu}$$

A necessary condition for the existence of the inverse transformation is, of course, that the Jacobians of the transformations are not singular, ie $\det(\Lambda^{\mu'}{}_{\mu}) \neq 0$; $\det(\Lambda^{\mu}{}_{\mu'}) \neq 0$

Next we can consider how the gradient of a function varies after a coordinate transformation. To this scope, consider a scalar function ϕ associated to a given event p of $\mathcal{M}$, ie, consider a mapping between the set of events in $\mathcal{M}$ to a scalar function

$$p \in \mathcal{M} \longrightarrow \phi(x^\mu(p))$$

where $\{x^\mu\}$ is a possible choice of coordinates to represent p



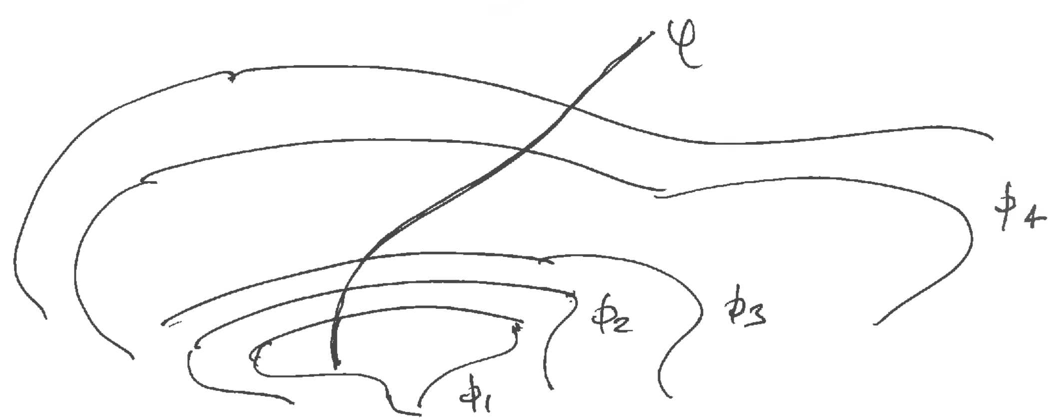
pressure field in meteorological map

To establish the variations of ϕ we need some direction along which to compute such variation, eg a curve $\mathcal{C}(\lambda)$ so that the variation of ϕ along $\mathcal{C}$ is

$$\frac{d\phi}{d\lambda} = \frac{\partial\phi}{\partial x^\mu}\frac{\partial x^\mu}{\partial\lambda} = \frac{\partial \phi}{\partial x^\mu} V^\mu =: C_\mu V^\mu$$

V^μ: tg vector to $\mathcal{C}$

where we have defined



$$\boxed{U_\mu := \frac{\partial \phi}{\partial x^\mu} := (\tilde{\partial}\phi)_\mu}$$

as the definition of the gradient of ϕ along ℓ

Note the position of index μ, which is different from the one used for the tangent vector $\underline{V}$ defined before. This is because U_μ is an object that follow a different transformation law under coordinate transformation

$$U_{\mu'} = \frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial \phi}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\mu'}} = U_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}$$

or equivalently

(2) $$\boxed{(\tilde{\partial}\phi)_{\mu'} = (\tilde{\partial}\phi)_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}}$$

cf.

(1) $$V^{\mu'} = V^\mu \frac{\partial x^{\mu'}}{\partial x^\mu}$$

(1) : trasformation law for a contravariant vector (or vector)
(2) : " " " " covariant vector (or covector, or one-form)

Vectors and covectors are elements of dual spaces ie the space of all covectors is a dual space to that of all tangent vectors

Note : a covector transforms differently from a coordinate transformation.

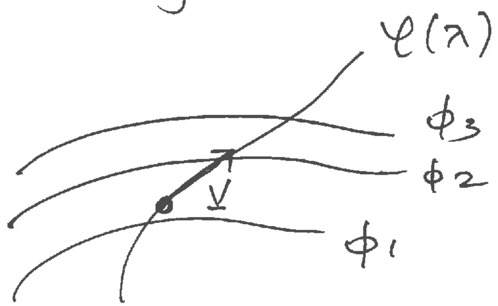
We have seen that

$$X^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} X^{\mu} \quad \text{and} \quad (\tilde{d}\phi)_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} (\tilde{d}\phi)_{\mu}$$

$$= \Lambda^{\mu'}{}_{\mu} X^{\mu} \quad \text{and} \quad = \Lambda^{\mu}{}_{\mu'} (\tilde{d}\phi)_{\mu}$$

similar notation but different objects

A geometrical view of vectors and covectors



We have seen that

$$\frac{d\phi}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial\phi}{\partial x^\mu} = V^\mu \left(\tilde{d}\phi\right)_\mu$$

Hence $\underline{V}$ as a geometrical object (ie not seen through its local components) should be seen as the tangent to a curve $\mathcal{C}(\lambda)$ and measuring how the scalar function ϕ changes along $\mathcal{C}$.

As a geometrical object, the vector $\underline{V}$ actually defines the curve to which it is locally tangent.

$$V^\mu = \begin{pmatrix}\text{contravariant} \\ \text{components}\end{pmatrix} = \begin{pmatrix}\text{measure} \\ \text{change of} \\ x^\mu \text{ along } \mathcal{C}\end{pmatrix}$$

V^μ will change with different coordinates but $\underline{V}$ will not of course we don't need the scalar function ϕ and can simply write

$$\underline{V} := \frac{d}{d\lambda} = V^\mu \left(\frac{\partial}{\partial x^\mu}\right) = V^\mu \underline{e}_\mu = V^0 \underline{e}_0 + V^1 \underline{e}_1 + V^2 \underline{e}_2 + V^3 \underline{e}_3$$

where

$$\underline{e}_\mu := \left(\frac{\partial}{\partial x^\mu}\right) = \partial_\mu \qquad : \underline{\text{basis vectors}}$$

The lower index in $\{\underline{e}_\mu\}$ is not meant as a component but to distinguish the 4 basis vectors, e.g.

$$\underline{e}_1 = \{e^\mu_1\} = (e^0_1, e^1_1, e^2_1, e^3_1) = (0, 1, 0, 0)$$

$$\underline{e}_3 = \{e^\mu_3\} = (e^0_3, e^1_3, e^2_3, e^3_3) = (0, 0, 0, 1)$$

$\vdots$

unit basis vectors

Note that the notation $\underline{V} = V^\mu \partial_\mu$ should not be source of confusion as this is something you are accustomed to from vector calculus in three space

$$\vec{V} = V^i \underline{e}_i = V^x \underline{e}_x + V^y \underline{e}_y + V^z \underline{e}_z$$

$$= V^r \underline{e}_r + V^\theta \underline{e}_\theta + V^\phi \underline{e}_\phi$$

$$= \dots$$

In a similar way, we can build a geometrical view of a covector $\underset{\sim}{d}$ that acts on vectors to produce a real number. in other words, given a vector $\underline{V}$, the action of $\underset{\sim}{d}$ on $\underline{V}$ yields a scalar, ie

$$\underset{\sim}{d}(\underline{V}) \overset{\overset{d_0 V^0 + d_1 V^1 + d_2 V^2 + d_3 V^3}{=}}{=} d_\mu V^\mu \in \mathbb{R}$$

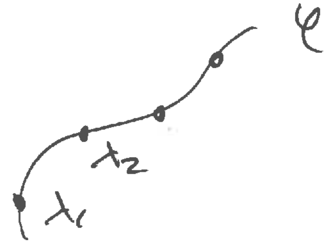
Note that this should not be confused with $\underline{V} = V^\mu \underline{e}_\mu$ while $V^\mu d_\mu$ is a $\underline{\text{scalar}}$ $\underline{\text{quantity}}$ (ie a real number!)

$\uparrow$ vectors

Recap

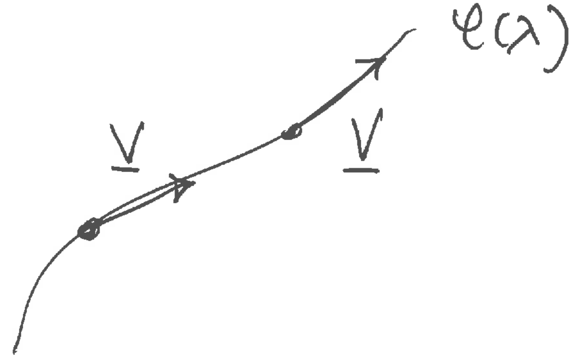
- A sequence of events selects a path (geometrical object). A curve is a parametrization of a path in a given coordinate system

$$\mathcal{C}(\lambda) : \{ x^\mu(\lambda), \text{ with } \lambda \in I \subset \mathbb{R} \}$$



- tangent vector : measures the change of coordinates along $\mathcal{C}(\lambda)$

$$V^\mu := \frac{dx^\mu}{d\lambda} \quad : \text{ components of } \underline{V} \text{ (geom. object)}$$



V^μ will depend on coord. system. $\underline{V}$ is intrinsic object

- $V^{\mu'} = \Lambda^{\mu'}{}_{\mu} V^{\mu}$ $\qquad \Lambda^{\mu'}{}_{\mu} := \dfrac{\partial x^{\mu'}}{\partial x^{\mu}}$: transformation matrix

  dummy (connected) indices

$\{x^\mu\} \to \{x^{\mu'}\}$

- $\{x^\mu\} \to \{x^{\mu'}\} \to \{x^{\mu''}\}$

  $V^{\mu''} = \Lambda^{\mu''}{}_{\mu'} \Lambda^{\mu'}{}_{\mu} V^{\mu}$ $\qquad \mu''$: free index

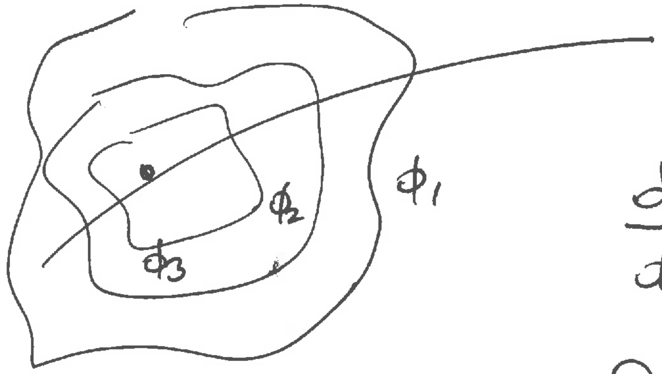
- inverse transformation

  $V^{\mu} = \Lambda^{\mu}{}_{\mu'} V^{\mu'}$ $\qquad \Lambda^{\mu}{}_{\mu'} := \dfrac{\partial x^{\mu}}{\partial x^{\mu'}}$

  $\Lambda^{\mu}{}_{\mu'} \Lambda^{\nu'}{}_{\nu} = \delta^{\mu}_{\nu}$: Kronecker delta

where $\det(\Lambda^{\mu'}{}_{\mu}) \neq 0$; $\det(\Lambda^{\mu}{}_{\mu'}) \neq 0$

- $P \in \mathcal{M}$ $\qquad \phi(x^\mu(P))$ scalar function in P



$$\frac{d\phi}{d\lambda} = \frac{\partial \phi}{\partial x^\mu}\frac{\partial x^\mu}{\partial \lambda} = \frac{\partial \phi}{\partial x^\mu} V^\mu = u_\mu V^\mu$$

$$u_\mu := \frac{\partial \phi}{\partial x^\mu} := (\tilde{d}\phi)_\mu \qquad \text{gradient of } \phi$$

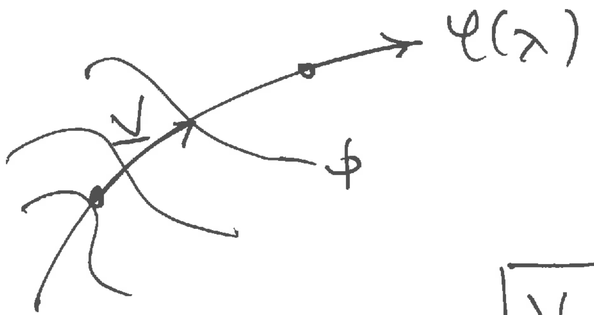
- $$u_{\mu'} = \frac{\partial \phi}{\partial x^{\mu'}} = \frac{\partial \phi}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\mu'}} = u_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}$$

$$(\tilde{d}\phi)_{\mu'} = (\tilde{d}\phi)_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}$$

$$= \Lambda^\mu{}_{\mu'} (\tilde{d}\phi)_\mu$$

: transformation law for a covector; this is a dual to a vector (more later)

• $\underline{V}$ as a geometrical object selects the direction of the curve to which it is tangent



$$\frac{d\phi}{d\lambda} = V^\mu (\tilde{\partial}\phi)_\mu = \frac{\partial\phi}{\partial x^\mu}\frac{\partial x^\mu}{\partial \lambda} = \frac{\partial x^\mu}{\partial\lambda}\frac{\partial\phi}{\partial x^\mu}$$

$$\boxed{\underline{V} := \frac{d}{d\lambda}} = V^\mu \underline{\partial}_\mu = V^\mu \underline{e}_\mu$$

$\{\underline{e}_\mu\}$ 4 basis vectors

$$\underline{e}_\mu := \frac{\partial}{\partial x^\mu}$$

$\underline{V} = V^\mu \underline{\partial}_\mu = V^\mu \underline{e}_\mu$ is familiar from vector algebra

Covectors are linear operators, ie

$$\underset{\sim}{d} = a\underset{\sim}{p} + b\underset{\sim}{q} \quad ; \quad \underset{\sim}{d}(\underline{V}) = a\underset{\sim}{p}(\underline{V}) + b\underset{\sim}{q}(\underline{V})$$

Similarly $\underset{\sim}{d}$ are linear in their arguments, ie

$$\underset{\sim}{d}(a\underline{V} + b\underline{U}) = a\underset{\sim}{d}(\underline{V}) + b\underset{\sim}{d}(\underline{U})$$

Note that both d_μ and V^μ, that is, both the covariant and contravariant components depend on the choice made for the coordinates. However the action of $\underset{\sim}{d}$ on $\underline{V}$ is coordinate independent.

Let's prove this important result

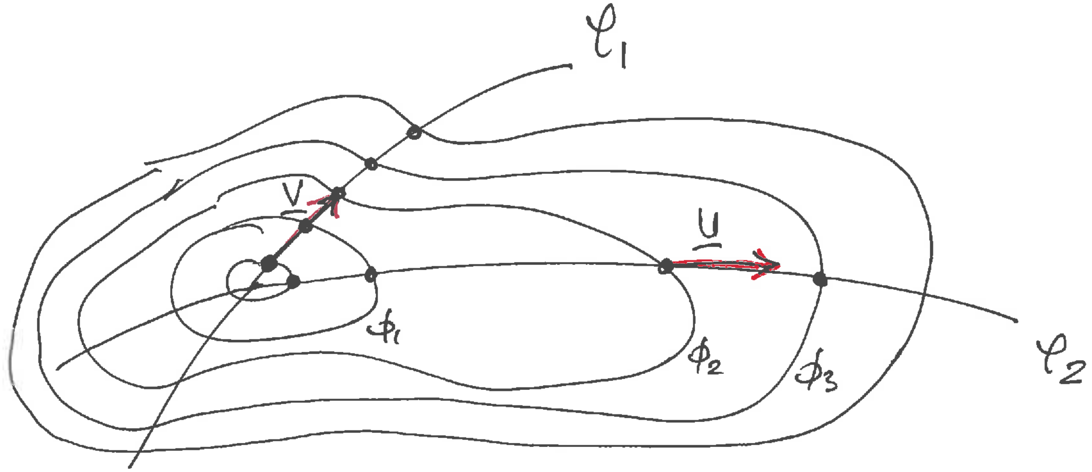
$$\underset{\sim}{d}(\underline{V}) = d_{\mu'} V^{\mu'} = (\Lambda^{\mu}{}_{\mu'} d_\mu)(\Lambda^{\mu'}{}_{\nu} V^{\nu})$$

$$= \Lambda^{\mu}{}_{\mu'} \Lambda^{\mu'}{}_{\nu} V^{\nu} d_\mu = \delta^{\mu}{}_{\nu} V^{\nu} d_\mu = d_\mu V^\mu$$

To understand the geometrical meaning of a covector let's bear in mind that

$$\tilde{\underline{d}}(\underline{V}) = x \in \mathbb{R}$$

: ie contracting a vector with a covector leads to a number



$$\tilde{\underline{d}}(\underline{V}) = \frac{\partial \phi}{\partial x^\mu} \frac{\partial x^\mu}{\partial \lambda}\Big|_{\text{along } \mathcal{C}_1}$$

$$\tilde{\underline{d}}(\underline{U}) = \frac{\partial \phi}{\partial x^\mu} \frac{\partial x^\mu}{\partial \lambda}\Big|_{\text{along } \mathcal{C}_2}$$

In other words, $\tilde{\underline{d}}(\underline{V})$ will measure the number of constant-ϕ surfaces pierced by $\underline{V}$ along $\mathcal{C}_1$. The longer the gradient of ϕ, the longer the number of surfaces pierced by the vector

$$\underset{[5]}{\tilde{\underline{d}}(\underline{V})} > \underset{[4]}{\tilde{\underline{d}}(\underline{U})}$$

We can now study how covectors transfor after a coordinate transformation. We have seen that $\underline{V}$ is an invariant object

$$\underline{V} = V^{\mu} \underline{e}_{\mu} = V^{\mu'} \underline{e}_{\mu'} = \Lambda^{\mu}{}_{\nu'} V^{\nu'} \underline{e}_{\mu} = \Lambda^{\mu}{}_{\mu'} V^{\mu'} \underline{e}_{\mu} \Rightarrow$$

$$V^{\mu'} \left(\underline{e}_{\mu'} - \Lambda^{\mu}{}_{\mu'} \underline{e}_{\mu} \right) = 0 \quad \underset{V^{\mu'} \text{ generic}}{\Longrightarrow} \quad \boxed{\underline{e}_{\mu'} = \Lambda^{\mu}{}_{\mu'} \underline{e}_{\mu}} \qquad (*)$$

Note that (*) is <u>not</u> a component transformation although (*) is clearly similar to the transformation of the components of a covector, ie

$$(\tilde{d}\phi)_{\mu'} = \Lambda^{\mu}{}_{\mu'} (\tilde{d}\phi)_{\mu}$$

From this one deduces that basis vectors transform as the components of covectors and the "opposite way" as the components of vectors.

Just as there are basis vectors, so there are basis covectors as the dual to the basis vectors.

$$\underset{\sim}{\tilde{\omega}}^{\mu} : \text{basis covectors (4 different ones)}$$

such that

$$\underset{\sim}{\tilde{\omega}}^{\mu}(\underline{e}_{\nu}) = \delta^{\mu}{}_{\nu}$$

eg

$$\underset{\sim}{\tilde{\omega}}^{0}(\underline{e}_{\nu}) = (\tilde{\omega}^{0})_{\alpha}(\underline{e}_{\nu})^{\alpha}$$

$$\Big|_{\nu=1}$$

$$= (\tilde{\omega}^{0})_{\alpha}(\underline{e}_{1})^{\alpha}$$

$$= (\tilde{\omega}^{0})_{0}\cancel{(\underline{e}_{1})^{0}} + \cancel{(\tilde{\omega}^{0})_{1}}(\underline{e}_{1})^{1} + \cancel{(\tilde{\omega}^{0})_{2}}\cancel{(\underline{e}_{1})^{2}} + \cancel{(\tilde{\omega}^{0})_{3}}\cancel{(\underline{e}_{1})^{3}}$$

$$= 0$$

$$\tilde{\underline{\omega}}^2(\underline{e}_2) = \cancel{(\tilde{\omega}^2_0)}\cancel{(\underline{e}_2)}^0 + \cancel{(\tilde{\omega}^2_1)}\cancel{(\underline{e}_2)}^1 + (\tilde{\omega}^2_2)(\underline{e}_2)^2 + \cancel{(\tilde{\omega}^2_3)}\cancel{(\underline{e}_2)}^3$$

$$= 1 \cdot 1 = 1$$

$\square$

Because $\{\tilde{\underline{\omega}}^\mu\}$ are the basis covectors, any covector can be expressed in their expansion, ie

$$\tilde{\underline{P}} = p_\mu \, \tilde{\underline{\omega}}^\mu$$

$\square$

This has the important consequence that we can now prove that $\tilde{\underline{\omega}}^\mu(\underline{e}_\nu) \overset{?}{=} \delta^\mu{}_\nu$

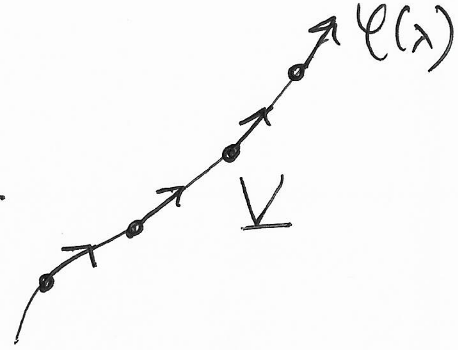
$$p_\mu V^\mu = \tilde{P}(\underline{V}) = p_\mu \tilde{\underline{\omega}}^\mu(\underline{V})$$

$$= p_\mu \tilde{\underline{\omega}}^\mu(V^\nu \underline{e}_\nu) = p_\mu V^\nu \tilde{\underline{\omega}}^\mu(\underline{e}_\nu)$$

$$\Rightarrow \quad p_\mu \left(V^\mu - V^\nu \underline{\tilde{\omega}}^\mu (\underline{e}_\nu) \right) = 0 \Leftrightarrow$$

$$\underline{\tilde{\omega}}^\mu (\underline{e}_\nu) = \delta^\mu{}_\nu \qquad \text{qed.}$$

Recap

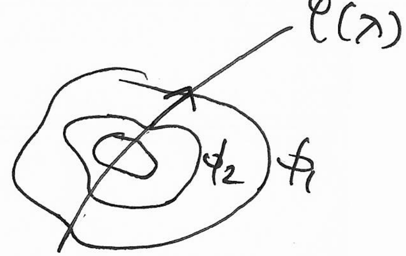


- vectors define curve to which they are tangent

$$\underline{V} := \frac{d}{d\lambda} = \frac{\partial\phi}{\partial x^\mu}\frac{\partial x^\mu}{\partial\lambda} = V^\mu \underline{e}_\mu = V^\mu \underline{\partial}_\mu$$

$\underline{e}_\mu$: basis vectors : 4 different 4-vectors.

- covectors are dual to vectors and are defined via a scalar function of which they measure the gradient.



Covectors need to be applied to a vector since the gradient needs to be defined along a certain direction

$$\tilde{d}(\underline{V}) = x \in \mathbb{R}$$

the action of a covector on a vector leads to a scalar (number)

The scalar measures the number of const values surfaces of a scalar function in the direction singled out by $\underline{V}$

- Vectors and covectors can be expressed in terms of components (contra and covariant) and basis vector and covectors

$$\underline{V} = V^\mu \underset{\sim}{e}_\mu \quad ; \quad \underline{\tilde{p}} = p_\mu \underline{\tilde{\omega}}^\mu$$

basis covectors

$$\underset{\sim}{e}_\mu (\underline{\tilde{\omega}}^\nu) = \delta^\nu_\mu = \underline{\tilde{\omega}}^\nu (\underset{\sim}{e}_\mu)$$

- Basis vectors and covectors follow different transformation laws

$$\underset{\sim}{e}_{\mu'} = \Lambda^\mu{}_{\mu'} \underset{\sim}{e}_\mu \quad ; \quad \underline{\tilde{\omega}}^{\mu'} = \Lambda^{\mu'}{}_\mu \underline{\tilde{\omega}}^\mu$$

In other words, basis vectors transform like the components of covectors

basis covectors " " " " of vectors

Note that all of the concepts presented so far have referred to a vector / covector as geometrical objects defined at a point (event) of the manifold M. However, these concepts can be extended to any point of M, thus introducing the concept of a <u>vector field</u> such that

$$\{ P \in M \longrightarrow V^\mu(P) \}$$

V^μ are the components of the vector field $\underline{V}$ at P in the coordinate system $\{x^\mu\}$.

The field is <u>differentiable</u> if V^μ are differentiable functions

The introduction of vectors and covectors has helped to define the simplest objects of a <u>larger space</u>, that of <u>tensors</u>. These can also be seen as entirely <u>geometric</u>

objects that are defined through the laws under which they transform when subject to a coordinate transformation.

For example, we can define a contravariant tensor of rank 2 (or tensor of contravariant rank 2) as a geometrical object $\underline{T}$ whose component change according to the transformation rule

$$T^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} T^{\mu\nu} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} T^{\mu\nu} \qquad (1)$$

Similarly, we can define a covariant tensor of rank 2 an object $\underline{T}$ whose components transform as

$$T_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} T_{\mu\nu} \qquad (2)$$

If the coordinates are regular (ie with nonzero Jacobian), then (1) and (2) will admit inverse transformations, ie

$$T^{\mu\nu} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} T^{\mu'\nu'}$$

$$T_{\mu\nu} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} T_{\mu'\nu'} .$$

Of course we can define covariant and contravariant tensors of any rank and even <u>mixed tensors</u>, ie geometrical objects having <u>both</u> contravariant and covariant components.

Example: we can define a tensor of contravariant rank 4 and covariant rank 2, (or mixed tensor of "type" (4,2)) an object whose components transform as

$$R^{\alpha'\beta'\gamma'\delta'}{}_{\mu'\nu'} = \Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta'}{}_{\beta}\Lambda^{\gamma'}{}_{\gamma}\Lambda^{\delta'}{}_{\delta}\Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'} R^{\alpha\beta\gamma\delta}{}_{\mu\nu}$$

or, under an inverse coordinate transformation, will transform as

$$R^{\alpha\beta\gamma\delta}{}_{\mu\nu} = \Lambda^{\alpha}{}_{\alpha'}\Lambda^{\beta}{}_{\beta'}\Lambda^{\gamma}{}_{\gamma'}\Lambda^{\delta}{}_{\delta'}\Lambda^{\mu'}{}_{\mu}\Lambda^{\nu'}{}_{\nu} R^{\alpha'\beta'\gamma'\delta'}{}_{\mu'\nu'}$$

Clearly, there are no limits (apart those imposed by convenience) and we can define vector spaces $\underline{V}^m_n$ as the space of all tensors of rank $(4,2)$, eg

$$R^{\alpha\beta\gamma\delta}{}_{\mu\nu} \in \underline{V}^4_2$$

Within this more general framework a <u>vector</u> is a tensor of rank $(1,0)$, while a <u>covector</u> is a tensor of rank $(0,1)$.

If a vector can be expressed in terms of basis vectors and a covector can be expressed in terms of basis covectors

$$\underline{V} = V^{\mu} \underline{e}_{\mu} \; ; \quad \tilde{p} = p_{\mu} \underline{\tilde{\omega}}^{\mu}$$

a generic mixed tensor can be expressed in terms of basis vectors and covectors. Example

$\underline{R} \in \underline{V}^{1}_{1}$ (1, 1) rank

$$\underline{R} = R^{\alpha}{}_{\beta} \, \underline{e}_{\alpha} \otimes \underline{\tilde{\omega}}^{\beta}$$

Note that basis vector and covectors do not commute,

ie $\underline{e}_{\alpha} \otimes \underline{\tilde{\omega}}^{\beta} \neq \underline{\tilde{\omega}}^{\beta} \otimes \underline{e}_{\alpha}$

so that the two tensors $\underline{R} = R^{\alpha}{}_{\beta} \, \underline{e}_{\alpha} \otimes \underline{\tilde{\omega}}^{\beta}$ and $\tilde{\underline{R}} = R_{\beta}{}^{\alpha} \, \underline{\tilde{\omega}}^{\beta} \otimes \underline{e}_{\alpha}$ are geometrically distinct. This is why the position of the indices is important.

Just like we have defined a vector field, so we can define a tensor field by associating a tensor to a point $P \in \mathcal{M}$, ie

$$f : \{ P \in \mathcal{M} \longrightarrow V^{\alpha\beta\ldots}{}_{\mu\nu\ldots}(P) \}$$ where $\underline{V}$ is a tensor of rank (m, n).

Let D be the dimension of the manifold $\mathcal{M}$, then a tensor $\underline{T}$ with N indices has $\underline{D^N}$ components.

Eg 4-dimensional spacetime

$$\left.\begin{array}{llll} T^{\mu} & \text{has} & D^1 = 4 & \text{components} \\ T^{\mu\nu} & \text{"} & 4^2 = 16 & \text{"} \\ T^{\mu\nu\alpha} & \text{"} & 4^3 = 64 & \text{"} \\ T^{\alpha\beta}{}_{\mu\nu} & \text{"} & 4^4 = 256 & \text{"} \end{array}\right\} \text{Matrices} \quad \underbrace{D \times D \times \ldots D}_{N}$$

$\vdots$

Nomenclature

Different authors use different nomenclature for the same object

V^μ: contravariant vector; contravariant tensor of rank 1; tensor of type $(1,0)$; $\binom{1}{0}$ tensor.

V_μ: covariant vector; covariant tensor of rank 1; tensor of type $(0,1)$; $\binom{0}{1}$ tensor or one-form

$V_{\mu\nu}$: covariant tensor of rank 2; tensor of type $(0,2)$ $\binom{0}{2}$ tensor (or two-form if antisymmetric).

In what follow we will first learn the basics of tensor <u>algebra</u> and subsequently of tensor <u>calculus</u>

(i) zero tensor : is a tensor which has all component that are zero in one coordinate system and hence in all coord. systems.

(ii) identical tensors : if two tensors of the same type have all components that are identical in one coord. system they will maintain them identical in all coord. systems.

(iii) scalar function multiplication : the multiplication of a scalar function with a tensor will yield a tensor of the same type

$$\underline{X} \in \underline{V}^m_n \ , \quad \underline{Y} := \phi \underline{X} \ , \quad \text{then} \quad \underline{Y} \in \underline{V}^m_n$$

(iv) addition : the addition of two tensors of the same type yields a tensor of the same type

$$\underline{X}, \underline{Y} \in \underline{V}^m_n \ , \quad \underline{Z} := \underline{X} + \underline{Y} \quad \text{with} \quad \underline{Z} \in \underline{V}^m_n$$

(v) multiplication or outer product : the multiplication of two tensors of any type yields a new tensor whose type is the sum of the two:

$\underline{X} \in \underline{V}^m_n$, $\underline{Y} \in \underline{V}^p_q$ and $\underline{Z} := \underline{X} \otimes \underline{Y}$, then $\underline{Z} \in \underline{V}^{m+p}_{n+q}$

Ex.

$$Z^{\alpha\beta\gamma}{}_{\lambda\mu\nu} = X^{\alpha\beta}{}_{\lambda}\, Y^{\gamma}{}_{\mu\nu}$$

(vi) <u>contraction</u>: contraction of a pair of indices in a mixed tensor of type (m,n) yields a tensor of type $(m-1, n-1)$, ie

$$Z^{\alpha\beta\mu}{}_{\mu\gamma\delta} = Z^{\alpha\beta}{}_{\gamma\delta}$$

(vii) <u>symmetries</u> and <u>antisymmetries</u>: a tensor of type (m,n) is symmetric on any pair of indices p, q if the components do not change after exchange of these indices. Conversely, the tensor will change its sign if antisymmetric

$$Z_{\mu\nu} \text{ symmetric } \iff Z_{\mu\nu} = Z_{\nu\mu}$$

$$Z_{\mu\nu} \text{ antisymmetric } \iff Z_{\mu\nu} = -Z_{\nu\mu}$$

Note: 1) given a generic tensor it is always possible to construct a symmetric or antisymmetric form by suitably combining its components

$$Z_{(\mu\nu)} := \frac{1}{2}\left(Z_{\mu\nu} + Z_{\nu\mu}\right) \quad : \text{ symmetric}$$

$$Z_{[\mu\nu]} = \frac{1}{2}\left(Z_{\mu\nu} - Z_{\nu\mu}\right) \quad : \text{ antisymmetric}$$

If D is the dimension of the manifold, a symmetric rank 2 tensor will have $\frac{1}{2}D(D+1)$ independent components and $\frac{1}{2}D(D-1)$ is antisymmetric; This is 10 and 6 in $D=4$

2) the contraction of a symmetric and antisym. tensor is zero

$\underline{Z}$ sym ; $\underline{V}$ antisym. $\qquad Z_{\mu\nu} V^{\mu\nu} = 0$

$$Z_{\mu\nu} V^{\mu\nu} = Z_{\nu\mu}\left(-V^{\nu\mu}\right) = -Z_{\nu\mu} V^{\nu\mu} = -Z_{\mu\nu} V^{\mu\nu}$$

μ, ν are dummy indices

$$\Rightarrow 2 Z_{\mu\nu} V^{\mu\nu} = 0 \qquad \text{qed.}$$

3) An arbitrary tensor can be decomposed in a symmetric and antisymmetric part

$$Z_{\mu\nu} = Z_{(\mu\nu)} + Z_{[\mu\nu]}$$

4) symmetization/antisym. for higher rank tensors proceeds in the same manner, eg

$$Z_{(\mu\nu\rho)} = \frac{1}{3!}\left(Z_{\mu\nu\rho} + Z_{\nu\mu\rho} + Z_{\rho\mu\nu} + Z_{\mu\rho\nu} + Z_{\nu\rho\mu} + Z_{\rho\nu\mu}\right)$$

$$Z_{[\mu\nu\rho]} = \frac{1}{3!}\left(Z_{\mu\nu\rho} - Z_{\nu\mu\rho} + Z_{\rho\mu\nu} - Z_{\mu\rho\nu} + Z_{\nu\rho\mu} - Z_{\rho\nu\mu}\right).$$

5) We can now introduce the Levi-Civita tensor, a generalization to four dimensions of the Levi-Civita symbol ϵ_{ijk}.

6) Any antisymmetric tensor of rank 5 or longer is zero. Why?

$$\epsilon_{\alpha\beta\gamma\delta} := -\sqrt{-g}\,\eta_{\alpha\beta\gamma\delta}$$

$$\epsilon^{\alpha\beta\gamma\delta} := \frac{1}{\sqrt{-g}}\,\eta^{\alpha\beta\gamma\delta}$$

where $\underline{g}$ tensor of rank $(0,2)$ and η is the totally antisym. metric symbol, such that

$$\eta_{\alpha\beta\gamma\delta} := \begin{cases} +1 & \text{if } [\alpha\beta\gamma\delta] \text{ even permutation of } 0123 \\ -1 & \text{if } [\alpha\beta\gamma\delta] \text{ odd } \quad '' \quad\quad '' \ 0123 \\ 0 & \text{if } [\alpha\beta\gamma\delta] \text{ are all different} \end{cases}$$

Ex

$$\eta_{0123} = \eta_{3012} = \eta_{2301} = \eta_{1230} = 1$$

$$\eta_{1023} = \eta_{0132} = -1 = \cdots$$

Note that $\eta_{\alpha\beta\gamma\delta}$ is <u>not</u> a tensor; its indices are not covariant or contravariant; they are simply a sequence of numbers

The contraction of two Levi-Civita tensors leads to additional permutation tensors

$$\left.\begin{aligned} \epsilon^{\alpha\beta\gamma\delta}\,\epsilon_{\lambda\mu\nu\delta} &= -1!\,\delta^{\alpha\beta\gamma}_{\lambda\mu\nu} \\ \epsilon^{\alpha\beta\gamma\delta}\,\epsilon_{\lambda\mu\gamma\delta} &= -2!\,\delta^{\alpha\beta}_{\lambda\mu} \end{aligned}\right\} \longrightarrow \text{extensions of Kronecker delta } \delta^{\alpha}_{\beta}$$

$$\epsilon^{\alpha\beta\gamma\delta}\,\epsilon_{\lambda\beta\gamma\delta} = -3!\,\delta^{\alpha}_{\lambda}$$

$$\epsilon^{\alpha\beta\gamma\delta}\,\epsilon_{\alpha\beta\gamma\delta} = -4!$$

$$\delta^{\alpha\beta}_{\lambda\mu} = \begin{vmatrix} \delta^{\alpha}_{\lambda} & \delta^{\alpha}_{\mu} \\ \delta^{\beta}_{\lambda} & \delta^{\beta}_{\mu} \end{vmatrix} = \delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu} - \delta^{\alpha}_{\mu}\delta^{\beta}_{\lambda} = 2\delta^{\alpha}_{[\lambda}\delta^{\beta}_{\mu]}$$

$$\delta^{\alpha\beta\gamma}_{\lambda\mu\nu} = \begin{vmatrix} \delta^{\alpha}_{\lambda} & \delta^{\alpha}_{\mu} & \delta^{\alpha}_{\nu} \\ \delta^{\beta}_{\lambda} & \delta^{\beta}_{\mu} & \delta^{\beta}_{\nu} \\ \delta^{\gamma}_{\lambda} & \delta^{\gamma}_{\mu} & \delta^{\gamma}_{\nu} \end{vmatrix} = \delta^{\alpha}_{\lambda}\left(\delta^{\beta}_{\mu}\delta^{\gamma}_{\nu} - \delta^{\beta}_{\nu}\delta^{\gamma}_{\mu}\right) - \delta^{\alpha}_{\mu}\left(\delta^{\beta}_{\lambda}\delta^{\gamma}_{\nu} - \delta^{\beta}_{\nu}\delta^{\gamma}_{\lambda}\right) + \delta^{\alpha}_{\nu}\left(\delta^{\beta}_{\lambda}\delta^{\gamma}_{\mu} - \delta^{\beta}_{\mu}\delta^{\gamma}_{\lambda}\right)$$

$\epsilon^{\alpha\beta\gamma\delta}$ is extension of Levi-Civita symbol ϵ_{ijk}

Recall that if

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$B^i = \epsilon^{ijk} \partial_j A_k \quad \Longrightarrow$$

$$B^x = \epsilon^{123} \partial_2 A_3 + \epsilon^{132} \partial_3 A_2$$

$$= \partial_2 A_3 - \partial_3 A_2 = \partial_y A^z - \partial_z A^y$$

$\vdots$

$$B^z = \epsilon^{312} \partial_1 A_2 + \epsilon^{321} \partial_2 A_1$$

$$= \partial_x A^y - \partial_y A^x$$

7) Tensor equations : these involve tensors and are invariant under coordinate transformations by construction !

(·) $$\boxed{G^{\mu}{}_{\nu} = k\, T^{\mu}{}_{\nu}}$$ in $\{x^{\mu}\}$

The same equation in a new coordinate system $\{x^{\mu'}\}$ will be

$$\Lambda^{\mu}{}_{\mu'} \Lambda^{\nu'}{}_{\nu}\, G^{\mu'}{}_{\nu'} = k\, \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu'}{}_{\nu}\, T^{\mu'}{}_{\nu'} \iff$$

$$\Lambda^{\mu}{}_{\mu'} \Lambda^{\nu'}{}_{\nu} \left(G^{\mu'}{}_{\nu'} - k\, T^{\mu'}{}_{\nu'} \right) = 0 \implies$$

(··) $$\boxed{G^{\mu'}{}_{\nu'} = k\, T^{\mu'}{}_{\nu'}}$$ since Λ is generic transformation

(·) and (··) have the same form $\iff$ they are covariant tensor equations

Let's review the concept of covariant eqs with a well known example

$$(1) \quad \partial_t \vec{B} = \vec{\nabla} \times \vec{E}$$

$$(2) \quad \partial_t \vec{E} = \vec{\nabla} \times \vec{B} - 4\pi \vec{J}$$

$$(3) \quad \vec{\nabla} \cdot \vec{E} = 4\pi \rho_e$$

$$(4) \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$\left.\right\}$ Maxwell eqs.

Replacing in (1) $\quad \partial_t (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \times \vec{E} = \vec{\nabla} \times (\partial_t \vec{A}) \Rightarrow$

$$\vec{\nabla} \times (\vec{E} + \partial_t \vec{A}) = 0 \Rightarrow \vec{E} = -(\vec{\nabla}\phi + \partial_t \vec{A})$$

We can consider the gauge transformation in which we add a new potential to ϕ and a new vector to $\vec{A}$, ie

$$(5) \begin{cases} \phi \to \phi' = \phi + \partial_t \psi \\ \vec{A} \to \vec{A}' = \vec{A} - \vec{\nabla}\psi \end{cases} \Rightarrow \quad \vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} - \vec{\nabla}\psi)$$

$$= \vec{\nabla} \times \vec{A} - \vec{\nabla} \times (\vec{\nabla}\psi) = \vec{\nabla} \times \vec{A} = \vec{B}$$

Similarly

$$\vec{E}' = -\left(\vec{\nabla}\phi' + \partial_t \vec{A}'\right) = -\left(\vec{\nabla}\left(\phi + \partial_t \psi\right) + \partial_t\left(\vec{A} - \vec{\nabla}\psi\right)\right)$$

$$= -\left(\vec{\nabla}\phi + \partial_t \vec{A} + \underbrace{\vec{\nabla}(\partial_t \psi) - \partial_t(\vec{\nabla}\psi)}_{=0}\right)$$

partial derivatives commute

$$= -\left(\vec{\nabla}\phi + \partial_t \vec{A}\right) = \vec{E}$$

In other words the fields $\vec{E}$ and $\vec{B}$ are invariant under the transformations (5); ψ : gauge function.

Metric tensor

So far we have learnt how to characterize a vector as a geometrical object defining a certain direction. However we have not yet discussed how to compute the "length" or modulus of a vector, nor how to compute the scalar product between two vectors. For both operations we need to endow the manifold with a symmetric tensor of type $(0,2)$ $\underline{g}$ such that when acting on two generic vectors $\underline{V}$ and $\underline{U}$ yields a real number, ie

$$\boxed{\underline{g}(\underline{U}, \underline{V}) = g_{\mu\nu} U^\mu V^\nu} \quad (3) \qquad g_{\mu\nu} U^\mu V^\nu = x \in \mathbb{R}$$

(3) represents the scalar product between $\underline{U}$ and $\underline{V}$

$$\underline{g}(\underline{U}, \underline{V}) \in \mathbb{R} \quad ; \quad \underline{g}(\underline{U}, \underline{V}) = 0 \iff \underline{U} \perp \underline{V}$$

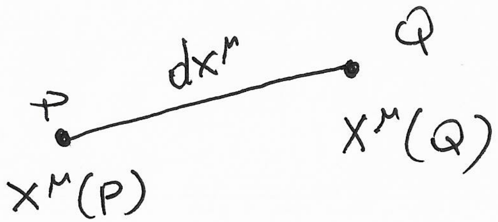
although $\underline{g}, \underline{U}, \underline{V}$ are coord. dependent!

Note that $\underline{g}(\underline{U}, \underline{V})$ is <u>coordinate independent</u> (Exercise)

It follows that the modulus of a vector is just the scalar product with itself, ie

$$\underline{g}(\underline{V}, \underline{V}) = g_{\mu\nu} V^\mu V^\nu = \underline{V} \cdot \underline{V} = \underline{V}^2 .$$

$\underline{g}$ can now be viewed as a <u>metric tensor</u>, ie a tensor measuring distances. Consider two events P and $Q \in M$ in a coordinate system $\{X^\mu\}$



We can evaluate the seperation

$$dx^\mu := X^\mu(Q) - X^\mu(P)$$

and consider dx^μ the components of a vector $\underline{dx}$ (note the whole $\underline{dx}$ is a vector and not $d\underline{x}$)

$$\begin{pmatrix} \text{distance between} \\ P \text{ and } Q \end{pmatrix} = \begin{pmatrix} \text{modulus of} \\ \text{4-vector } \underline{dx} \end{pmatrix} = \begin{pmatrix} \text{scalar product of} \\ \underline{dx} \text{ with itself} \end{pmatrix}$$

$$ds^2 := \underline{dx} \cdot \underline{dx} = \underline{g}\left(\underline{dx}, \underline{dx}\right) = g_{\mu\nu}\, dx^\mu dx^\nu = ds^2 \; : \text{ proper distance}$$

Of course ds^2 is coordinate independent since the relative (proper) distance between two events cannot depend on the coordinate system chosen: it's an invariant. Let's prove this important result

$$\begin{aligned} g_{\mu\nu}\, dx^\mu dx^\nu &= \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu}\, g_{\mu'\nu'}\, dx^\mu dx^\nu \\ &= \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu}\, g_{\mu'\nu'}\, \Lambda^{\mu}{}_{\alpha'}\, dx^{\alpha'}\, \Lambda^{\nu}{}_{\beta'}\, dx^{\beta'} \\ &= \underbrace{\Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} \Lambda^{\mu}{}_{\alpha'}}\, \Lambda^{\nu}{}_{\beta'}\, g_{\mu'\nu'}\, dx^{\alpha'} dx^{\beta'} \\ &= \delta^{\mu'}{}_{\alpha'}\, \delta^{\nu'}{}_{\beta'}\, g_{\mu'\nu'}\, dx^{\alpha'} dx^{\beta'} \\ &= g_{\mu'\nu'}\, \delta^{\mu'}{}_{\alpha'}\, \delta^{\nu'}{}_{\beta'}\, dx^{\alpha'} dx^{\beta'} = g_{\mu'\nu'}\, dx^{\mu'} dx^{\nu'} \quad \text{qed} \end{aligned}$$

Because $\underline{g}$ can be used to measure distances, it is also referred to as the <u>metric</u> tensor; in many respects, it is the most important tensor in general relativity.

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g_{0\nu}\, dx^0 dx^\nu + g_{1\nu}\, dx^1 dx^\nu + g_{2\nu}\, dx^2 dx^\nu + g_{3\nu}\, dx^3 dx^\nu$$

$$= g_{00} (dx^0)^2 + g_{01}\, dx^0 dx^1 + g_{02}\, dx^0 dx^2 + g_{03}\, dx^0 dx^3$$

$$+ g_{10}\, dx^1 dx^0 + g_{11} (dx^1)^2 + g_{12}\, dx^1 dx^2 + \dots$$

Note that $\underline{g}$ is necessarily a symmetric tensor, hence with $\frac{1}{2} D(D+1) = 10$ independent components.

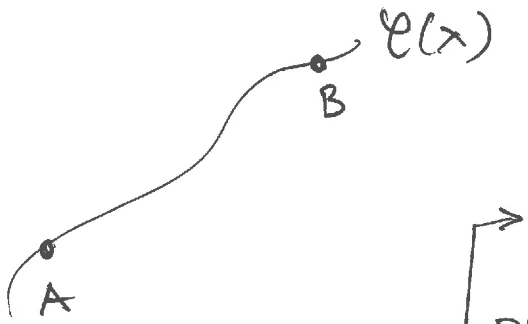
$$\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ & g_{11} & g_{12} & g_{13} \\ & & g_{22} & g_{23} \\ & & & g_{33} \end{pmatrix};$$

- the <u>signature</u> of $\underline{g}$ is the sum of the positive, negative, zero eigenvalues of $\underline{g}$

Ex

g: $(-,+,+,+)$: signature $+2$ } most common signatures
g: $(+,-,-,-)$: " -2 }

The metric tensor allows us to make <u>measurements</u> in the distance between events in a spacetime which is covered with a given coordinate system.



$$l_{AB} := \int_A^B \sqrt{ds^2} = \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \quad : \text{this is the extension of Pythagoras's rule}$$

$\hookrightarrow$ proper distance

Note that $\sqrt{ds^2} > 0$ but it is possible that $ds^2 < 0$ (more later).

Consider A, B in a three-dimensional space

$$d\ell^2 = dx^2 + dy^2 + dz^2 \qquad x^i = (x, y, z) : \text{Cartesian}$$

$$= d\rho^2 + \rho^2 d\phi^2 + dz^2 \qquad x^i = (\rho, \phi, z) : \text{cylindrical}$$

$$= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \qquad x^i = (r, \theta, \phi)$$

The three coordinate systems will have different metric tensors, ie

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Cartesian   cylindrical   spherical

spatial part of metric tensor

However, the distance $ds^2 = g_{ij}\, dx^i dx^j$ will be the same. EXERCISE

Note that $ds^2 = \sum_i (dx^i)^2$ only in Cartesian coordinates, which represent a very special case of metric tensor.

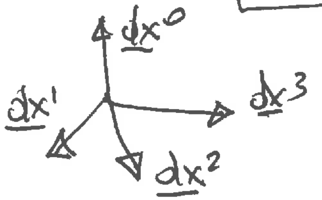
We can now go back to the Levi-Civita tensor and complete the discussion of its properties.

In particular, I recall that we have seen that

$$\epsilon_{\alpha\beta\mu\nu} = -\sqrt{-g}\;\eta_{\alpha\beta\gamma\delta}$$

where $g = \det(\underline{g})$, determinant of tensor of rank $(0,2)$, ie the metric tensor. To appreciate this result, consider a set of coordinates $\{x^\mu\}$ and compute the <u>volume element</u>

$$d^4x := dV = \left| \epsilon_{\alpha\beta\mu\nu}\, dx_0^\alpha\, dx_1^\beta\, dx_2^\mu\, dx_3^\nu \right| = \sqrt{-g}\;\eta_{\alpha\beta\mu\nu}\, dx_1^\alpha dx_0^\beta dx_2^\mu dx_3^\nu$$



$$= \sqrt{-g}\; d\tau$$

<u>proper volume element</u>

<u>coordinate volume element</u>

$$d\tau = dx_1 dx_2 dx_3 dx_0$$

More in general

$$V_p = \int_\Sigma \sqrt{-g}\, d^4x = \int dV = \int \sqrt{-g}\; d\tau$$

When $\underline{g}$ is non singular, it admits an inverse such that

$$\boxed{g^{\mu\nu} g_{\mu\lambda} = \delta^{\nu}{}_{\lambda}} \qquad (4)$$

This is an important property, that allows us to lower and raise indices in a tensor by contracting with $\underline{g}$ or it inverse

Ex

$$g_{\alpha\beta} T^{\alpha\delta}{}_{\mu\nu} = T_{\beta}{}^{\delta}{}_{\mu\nu} \quad ; \quad g^{\mu\beta} T^{\alpha\delta}{}_{\mu\nu} = T^{\alpha\delta\beta}{}_{\nu}$$

Note

$$g^{\mu\nu} Z_{\mu\nu} = Z^{\mu}{}_{\mu} = \mathrm{tr}\,(Z^{\mu\nu}) \qquad \text{if} \quad Z_{\mu\nu} = Z_{(\mu\nu)}$$

$$g^{\mu\nu} Z_{\mu\nu} = 0 \qquad \text{if} \quad Z_{\mu\nu} = Z_{[\mu\nu]}$$

The latter is a general rule: the contraction of a pair of symmetric indices with antisymmetric ones is zero.[1]

Mapping vectors into covectors and viceversa

$$\underline{g}(\underline{U}, \underline{V}) = \underline{U} \cdot \underline{V}$$

Consider now

$$\underline{g}(\underline{U}, \cdot) = \underline{g}(\cdot, \underline{U})$$

$$= \underline{\tilde{U}}(\cdot)$$

ie this is an operator defined in terms of $\underline{U}$ and acting on any vector $\underline{V}$ such that

$$\underline{\tilde{U}}(\underline{V}) = \underline{g}(\underline{U}, \underline{V}) = \underline{U} \cdot \underline{V} = g_{\mu\nu} U^{\mu} V^{\nu} = U_{\mu} V^{\mu}$$

In other words $\underline{g}$ maps a vector $\underline{U}$ into a covector $\underline{\tilde{U}}$

Similarly $\underline{g}^{-1}$ (the inverse of $\underline{g}$) maps a covector into a vector. This is just the duality between vectors and covectors.

Finally take the scalar product between two vectors $\underline{U}$, $\underline{V}$

$$\underline{U} \cdot \underline{V} = \underline{g}(\underline{U}, \underline{V}) = g_{\mu\nu} U^\mu V^\nu$$

$$= (U^\mu \underline{e}_\mu) \cdot (V^\nu \underline{e}_\nu)$$

$$= U^\mu V^\nu (\underline{e}_\mu \cdot \underline{e}_\nu) \Rightarrow$$

$$\boxed{g_{\mu\nu} = \underline{e}_\mu \cdot \underline{e}_\nu}$$

In other words, the components of the metric tensor represent the <u>scalar product of the basis vectors</u>

$\{\underline{e}_\mu\}$ is an <u>orthonormal</u> basis if $|\underline{e}_\mu \cdot \underline{e}_\nu| = |\delta_{\mu\nu}|$

Coordinate basis and orthonormal basis

We have seen that we can express a vector as

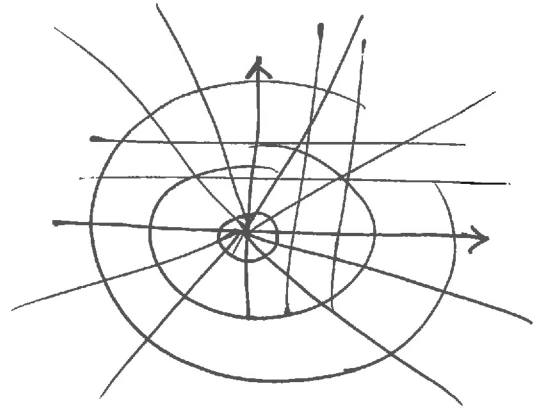
$$\underline{V} = V^{\mu} \partial_{\mu} = V^{\mu} \frac{\partial}{\partial x^{\mu}} = V^{\mu} \underline{e}_{\mu} \qquad \text{where } \underline{e}_{\mu} = \frac{\partial}{\partial x^{\mu}} \quad : \text{basis vectors}$$

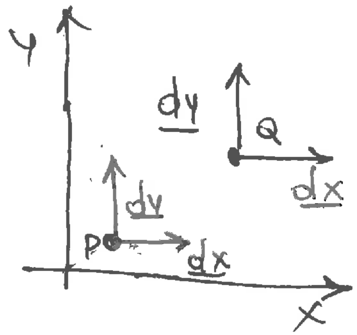
We have also seen that the metric tensor is the scalar product of basis vectors, $g_{\mu\nu} = \underline{e}_{\mu} \cdot \underline{e}_{\nu}$.

Clearly there are coordinate systems in which $g_{\mu\nu}$ has a particularly simple form and it is useful to express physical laws in terms of these coordinate basis.

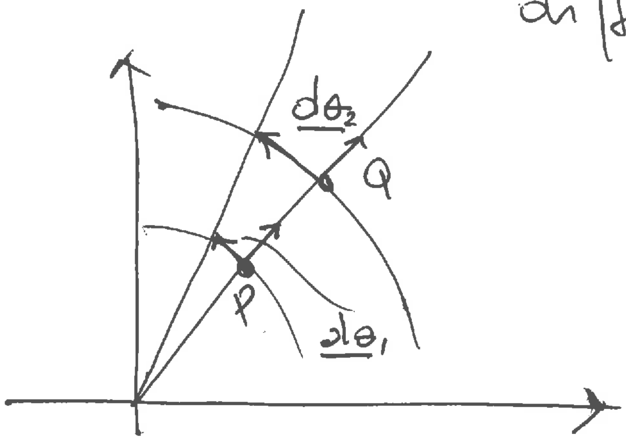
Let's make an example that will clarify the ideas. Let's consider an Euclidean plane covered either with Cartesian or with polar coordinates.

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$





Clearly $\underline{dx} = \underline{dy}$ and the same everywhere. On the other hand $\underline{d\theta}$ will be different at different radii



This information is indeed contained in the metric tensor

$$g_{ij} = \underline{e_i} \cdot \underline{e_j} = \left(\frac{\partial}{\partial x^i}\right)\left(\frac{\partial}{\partial x^j}\right) = \delta_{ij} \quad \text{in}$$

Cartesian coordinates

Ex

$$g_{xx} = \left(\frac{\partial}{\partial x}\right)\cdot\left(\frac{\partial}{\partial x}\right) = 1 \; ; \quad g_{xy} = \left(\frac{\partial}{\partial x}\right)\cdot\left(\frac{\partial}{\partial y}\right) = 0$$

(x, y) are independent coordinates

$$g_{yy} = \left(\frac{\partial}{\partial y}\right)\cdot\left(\frac{\partial}{\partial y}\right) = 1 \qquad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ;$$

On the other hand, in a polar coordinate system (r, θ)

$$g_{rr} = \underline{e}_r \cdot \underline{e}_r = \left(\frac{\partial}{\partial r}\right) \cdot \left(\frac{\partial}{\partial r}\right) = 1 \; ; \quad g_{r\theta} = \left(\frac{\partial}{\partial r}\right) \cdot \left(\frac{\partial}{\partial \theta}\right) = 0 \; ; \quad g_{\theta\theta} = \left(\frac{\partial}{\partial \theta}\right) \cdot \left(\frac{\partial}{\partial \theta}\right) = r^2$$

r, θ are
indep. coord. or
orthogonal coordinates

since

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$ is metric tensor in polar coordinates.

It's not difficult to define in this coordinate system a basis of vectors that is orthonormal, ie such that

$$\left| \underline{e}_{\hat{\alpha}} \cdot \underline{e}_{\hat{\beta}} \right| = \left| \delta_{\hat{\alpha}\hat{\beta}} \right|$$

All we need to do is to go from $\underline{e}_\mu = \frac{\partial}{\partial x^\mu}$ to

$$\underline{e}_{\hat{\mu}} = k \frac{\partial}{\partial x^\mu}$$

Ex.

$$\underline{e}_{\hat{r}} = \underline{e}_r \; ; \quad \underline{e}_{\hat{\theta}} = \frac{1}{r}\underline{e}_\theta$$

$$\underline{e}_{\hat{r}} \cdot \underline{e}_{\hat{r}} = \underline{e}_r \cdot \underline{e}_r = 1 \; ; \quad \underline{e}_{\hat{r}} \cdot \underline{e}_{\hat{\theta}} = \frac{1}{r}\underline{e}_r \cdot \underline{e}_\theta = 0$$

$$\underline{e}_{\hat{\theta}} \cdot \underline{e}_{\hat{\theta}} = \frac{1}{r^2}\underline{e}_\theta \cdot \underline{e}_\theta = \frac{r^2}{r^2} = 1$$

As a result, the basis $\underline{e}_{\hat{i}}$ is an <u>orthonormal</u> <u>basis</u>.

$$g_{\hat{i}\hat{j}} = \underline{e}_{\hat{i}} \cdot \underline{e}_{\hat{j}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

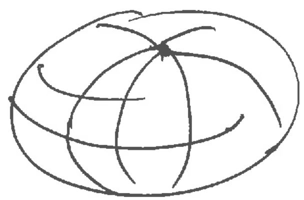
The advantage of an orthonormal basis is that the components of vector in this basis are directly related to the magnitude in that direction

$$\underline{V} = V^\mu \underline{e}_\mu = V^{\hat{\mu}} \underline{e}_{\hat{\mu}}$$

V^θ will depend on r

$V^{\hat{\theta}}$ does not depend on r

We can extend this example also to the (curved) surface of a 2-sphere



$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$g_{\theta\theta} = r^2 \; ; \quad g_{\phi\phi} = r^2 \sin^2\theta \; ; \qquad g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}$$

We can introduce an orthonormal basis

$$\underline{e}_{\hat\theta} = \frac{1}{r}\underline{e}_\theta \; ; \quad \underline{e}_{\hat\phi} = \frac{1}{r\sin\theta}\underline{e}_\phi \qquad \text{such that}$$

$$g_{\hat\theta\hat\theta} = \underline{e}_{\hat\theta}\cdot\underline{e}_{\hat\theta} = \frac{1}{r^2}\underline{e}_\theta\cdot\underline{e}_\theta = \frac{1}{r^2} g_{\theta\theta} = \frac{r^2}{r^2} = 1$$

$$g_{\hat\phi\hat\phi} = \dots\dots = \frac{r^2\sin^2\theta}{r^2\sin^2\theta} = 1 \qquad g_{\hat i\hat j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We can consider both a coordinate and an orthonormal basis as special cases of a "general basis"

Recap

- Given the Levi-Civita tensor $\epsilon_{\alpha\beta\mu\nu}$ we can define the proper volume element a $dV = |\epsilon_{\alpha\beta\mu\nu}\, dx_0^\alpha dx_1^\beta dx_2^\mu dx_3^\nu|$

$$= \sqrt{-g}\, \eta_{\alpha\beta\mu\nu}\, dx_0^\nu dx_1^\beta dx_2^\mu dx_3^\nu$$

$$V = \int dV = \int \sqrt{-g}\, \underbrace{d^4x}$$

← coordinate volume

- $g^{\mu\nu} g_{\mu\lambda} = \delta^\nu_\lambda \implies$ metric can be used to raise and lower the indices

$$g_{\alpha\beta} T^{\alpha\mu}{}_\delta = T_\beta{}^\mu{}_\delta \; , \quad g^{\alpha\beta} T_{\alpha\mu\nu} = T^\beta{}_{\mu\nu}$$

- $\underline{g}(\underline{u}, \underline{v}) = \underline{u} \cdot \underline{v} = g_{\mu\nu} u^\mu v^\nu \; ; \quad \underline{g}(\underline{u}, \cdot) = \underset{\sim}{\tilde{u}}(\cdot)$

$$\Rightarrow \quad \underline{g}(\underline{u}\cdot\underline{v}) = \underline{\tilde{u}}(\underline{v}) = \underline{u}\cdot\underline{v} = u_\mu v^\mu$$

g maps a vector into a covector $(\underline{u} \to \underline{\tilde{u}})$

- $g_{\mu\nu} = \underline{e}_\mu \cdot \underline{e}_\nu$: scaler product of basis vectors
- $\{\underline{e}_\mu\}$ is orthonormal if $|\underline{e}_\mu \cdot \underline{e}_\nu| = \delta_{\mu\nu}$
- $\underline{e}_\mu = \partial_\mu$: $\{\underline{e}_\mu\}$ is a coordinate basis
- if a coordinate basis is an orthonormal basis; metric is diagonal
- Not all basis are coordinate basis

$\{\underline{e}_x, \underline{e}_y\}$: coord. basis ; $\{\underline{e}_r, \underline{e}_\theta\}$: non-coord. basis

$\underline{e}_\theta \cdot \underline{e}_\theta = r^2 \neq 1$

• A non-coordinate basis can be written as an orthonormal basis after the use of suitable normalization factors

$\{\underline{e}_\mu\}$: non-coordinate basis $\Longleftrightarrow$ $|\underline{e}_\mu \cdot \underline{e}_\nu| \neq |\delta_{\mu\nu}|$

$\{\underline{e}_\mu\} \longrightarrow \{\underline{e}_{\hat{\mu}}\}$ where $\underline{e}_{\hat{\mu}} := k\, \underline{e}_\mu$ and k is chosen so that $|\underline{e}_{\hat{\mu}} \cdot \underline{e}_{\hat{\nu}}| = |\tilde{k}\, \underline{e}_\mu \cdot \underline{e}_\nu| = \delta_{\mu\nu}$

Ex

$\underline{e}_{\hat{r}} = \underline{e}_r$; $\quad \underline{e}_{\hat{\theta}} = \frac{1}{r} \underline{e}_\theta \qquad \underline{e}_{\hat{\theta}} \cdot \underline{e}_{\hat{\theta}} = \frac{1}{r^2}\, \underline{e}_\theta \cdot \underline{e}_\theta = \frac{r^2}{r^2} = 1$

$|g_{\hat{\mu}\hat{\nu}}| = |\underline{e}_{\hat{\mu}} \cdot \underline{e}_{\hat{\nu}}| = \delta_{\hat{\mu}\hat{\nu}}$

• $\underline{V} = V^\mu \underline{e}_\mu = V^{\hat{\mu}} \underline{e}_{\hat{\mu}}$ $\qquad$ in an orthonormal basis, $V^{\hat{\mu}}$ really represent the values of the vector in the direction $\hat{\mu}$ and do not depend on position

$$V^{\theta} \neq V^{\hat{\theta}} \quad ;$$

- a coordinate basis is an orthonormal basis and the corresponding metric is diagonal with elements 1 or 0

- a non-coordinate basis can either have a diagonal or a non-diagonal metric; a non-coordinate basis is also referred to as a <u>general basis</u>.

$\underline{e}_\alpha$: <u>general basis</u> $\boxed{\underline{e}_\alpha = \Lambda_\alpha{}^\mu \dfrac{\partial}{\partial x^\mu}}$ $= \Lambda_\alpha{}^\mu(x^\nu) \dfrac{\partial}{\partial x^\mu}$

Λ is function of x^ν

where $\underline{e}_\alpha$ is a <u>coordinate basis</u> if $\Lambda_\alpha{}^\mu = \delta^\mu{}_\alpha$ while

$\underline{e}_\alpha$ is an <u>orthonormal basis</u> if $\underline{e}_\alpha \cdot \underline{e}_\beta = \eta_{\alpha\beta}$

where $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1, 1) = \begin{pmatrix} -1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

ie $\eta_{ij} = \delta_{ij}$; $\eta_{00} = -1$

In this latter case the basis is also called a <u>tetrad</u> (vierbein). The advantages of an orthonormal basis are that all the complexities of the metric are transferred to the components.

To recap

$\underline{e}_\mu$ is generic basis

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \underline{e}_\mu \cdot \underline{e}_\nu\, dx^\mu dx^\nu = (\Lambda_\mu{}^{\bar\alpha}\, \underline{e}_{\bar\alpha})(\Lambda_\nu{}^{\bar\beta}\, \underline{e}_{\bar\beta})\, dx^\mu dx^\nu$$

$$= \left[\Lambda_\mu{}^{\bar\alpha}\, \underline{e}_{\bar\alpha}(\Lambda_\nu{}^{\bar\beta})\right] \underline{e}_{\bar\beta}\, dx^\mu dx^\nu + \underline{e}_{\bar\alpha} \cdot \underline{e}_{\bar\beta}\, \Lambda_\mu{}^{\bar\alpha} \Lambda_\nu{}^{\bar\beta}\, dx^\mu dx^\nu$$

1) $\{\underline{e}_\mu\}$ is coord. basis. $\Rightarrow$ Λ is unit matrix $\Rightarrow$ first term disappears because $\underline{e}_{\bar\alpha}(\Lambda_\nu{}^{\bar\beta}) = \partial_{\bar\alpha}(\Lambda_\nu{}^{\bar\beta}) = 0$

$\Lambda^{\bar\alpha}_\mu = \delta^{\bar\alpha}_\mu$ and so $ds^2 = \underline{e}_\mu \cdot \underline{e}_\nu\, dx^\mu dx^\nu$

2) $\{\underline{e}_\mu\}$ is orthonormal basis $\Rightarrow$ $|\underline{e}_{\bar\alpha} \cdot \underline{e}_{\bar\beta}| = \delta_{\bar\alpha\bar\beta}$

The first term will disappear only if metric is diagonal and the second term will be

$$\underline{e}_{\bar{\alpha}} \cdot \underline{e}_{\bar{\beta}} \, \Lambda_{\mu}{}^{\bar{\alpha}} \, \Lambda_{\nu}{}^{\bar{\beta}} \, dx^{\mu} dx^{\nu} = \underline{e}_{\hat{\alpha}} \cdot \underline{e}_{\hat{\beta}} \, \Lambda_{\mu}{}^{\hat{\alpha}} \, \Lambda_{\nu}{}^{\hat{\beta}} \, dx^{\mu} dx^{\nu}$$

$$= \underbrace{\underline{e}_{\hat{\alpha}} \cdot \underline{e}_{\hat{\beta}}}_{\text{simple metric}} \; \underbrace{dx^{\hat{\alpha}} \, dx^{\hat{\beta}}}$$

complex coordinate

where $\boxed{dx^{\hat{\alpha}} = \Lambda_{\mu}{}^{\hat{\alpha}} \, dx^{\mu}}$

Ex

$$ds^2 = dr^2 + r^2 d\theta^2 = d\hat{r}^2 + d\hat{\theta}^2$$

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} ; \quad g_{\hat{a}\hat{b}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This simplification is important and also has a physical significance as will become clearer later on in the course

How do we determine if a basis is a <u>coordinate basis</u>?

To this scope we can use the <u>commutator</u> of the basis vectors. More precisely, if $\underline{X}$, $\underline{Y}$ are two vector fields, we define the <u>commutator</u> of $\underline{X}$ with $\underline{Y}$ as ⊙

$$\underline{Z} := [\underline{X}, \underline{Y}] := \underline{X}\,\underline{Y} - \underline{Y}\,\underline{X}$$

(note order is important as it involves a sign change)

To calculate the components of $\underline{Z}$ we can employ a coordinate basis in which $\underline{X} = X^\mu \frac{\partial}{\partial x^\mu}$; $\underline{Y} = Y^\nu \frac{\partial}{\partial x^\nu}$

and consider the action of the commutator on a scalar function $\phi(x^\mu)$, ie

⊙ $[\underline{X}, \underline{Y}]$ are also called the Lie brackets of $\underline{X}$ and $\underline{Y}$

$$[\underline{X},\underline{Y}]\,\phi = \underline{X}(\underline{Y}(\phi)) - \underline{Y}(\underline{X}(\phi))$$

Because ϕ is arbitrary, we can choose $\phi = x^{\bar{\mu}}$ which is not meant to be a vector but a fixed choice of coordinate eg $\phi = x$ or $\phi = y$. In this case

$$\underline{X}(\underline{Y}(\phi)) = X^\mu \partial_\mu \left(Y^\nu \partial_\nu (x^{\bar{\mu}})\right) = X^\mu \partial_\mu Y^\nu \delta^{\bar{\mu}}_\nu = X^\mu \partial_\mu Y^{\bar{\mu}}$$

$$\underline{Y}(\underline{X}(\phi)) = Y^\mu \partial_\mu \left(X^\nu \partial_\nu (x^{\bar{\mu}})\right) = Y^\mu \partial_\mu X^{\bar{\mu}}$$

so that in general

$$\boxed{Z^\mu = [\underline{X},\underline{Y}]^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu}$$

If $\underline{e}_\mu$ is a <u>general basis</u>, then

$$[\underline{e}_\mu, \underline{e}_\nu] = c^\lambda_{\mu\nu}\, \underline{e}_\lambda$$

where $c^\lambda_{\mu\nu}$ are called the <u>structure coefficients</u>

$c^\lambda_{\mu\nu}$ is not a tensor as can be verified through a coordinate transformation.

It is not difficult to show that

$$c^\lambda_{\mu\nu} = (e^\alpha_\mu \partial_\alpha e^\beta_\nu - e^\alpha_\nu \partial_\alpha e^\beta_\nu)\, e^\lambda_\beta .$$

Proof

$$[\underline{e}_\mu, \underline{e}_\nu] = \underline{e}_\mu \underline{e}_\nu - \underline{e}_\nu \underline{e}_\mu$$

Apply the brackets to a function ϕ

$$[\underline{e}_\mu, \underline{e}_\nu]\phi = \underline{e}_\mu \underline{e}_\nu \phi - \underline{e}_\nu e_\mu \phi = \qquad \underline{e}_\mu = e^\alpha_\mu \partial_\alpha$$

$$= (e^\alpha_\mu \partial_\alpha)(e^\beta_\nu \partial_\beta \phi) - (e^\alpha_\nu \partial_\alpha)(e^\beta_\mu \partial_\beta \phi)$$

$$= (e^\alpha_\mu \partial_\alpha e^\beta_\nu)\partial_\beta \phi + e^\alpha_\mu e^\beta_\nu \partial_\alpha \partial_\beta \phi$$

$$-(e^\alpha_\nu \partial_\alpha e^\beta_\mu)\partial_\beta \phi - e^\alpha_\nu e^\beta_\mu \partial_\alpha \partial_\beta \phi$$

cancel because partial derivatives commute

$$= (e^\alpha_\mu \partial_\alpha e^\beta_\nu - e^\alpha_\nu \partial_\alpha e^\beta_\mu)\partial_\beta \phi \Rightarrow$$

$$[e_\mu, \underline{e}_\nu] = (e^\alpha_\mu \partial_\alpha e^\beta_\nu - e^\alpha_\nu \partial_\alpha e^\beta_\mu)\underline{e}_\beta$$

$$= C^\beta_{\mu\nu}\,\underline{e}_\beta \quad \text{qed} \quad \square$$

This brings the important result that vector basis is a coordinate basis if and only if all of its structure coefficients vanish.

In other words

$$[\underline{e}_\mu, \underline{e}_\nu] = 0 \iff \{\underline{e}_\mu\} \text{ is a } \underline{\text{coordinate basis}}$$

Proof

Let $\underline{e}_\mu = \partial_\mu$: coordinate basis

$$[\underline{e}_\mu, \underline{e}_\nu]^\alpha = e_\mu{}^\beta \partial_\beta e^\alpha_\nu - e^\beta_\nu \partial_\beta e^\alpha_\mu$$

$$= \delta^\beta_\mu \partial_\beta e^\alpha_\nu - \delta^\beta_\nu \partial_\beta e^\alpha_\mu$$

$$= \partial_\mu e^\alpha_\nu - \partial_\nu e^\alpha_\mu = \partial_\mu \delta^\alpha_\nu - \partial_\nu \delta^\alpha_\mu = 0 - 0 = 0.$$

Ex.

$$[\underline{e}_x, \underline{e}_y]^i = e_x{}^k \partial_k e^i_y - e^k_y \partial_k e^i_x = \delta^k_x \partial_k e^i_y - \delta^k_y \partial_y e^i_x$$

Let $i = 1$

$$= \partial_x (e^1_y) - \partial_y (e^1_x) = \underbrace{\partial_x(0) - \partial_y(1)}_{=0}$$

On the other hand,

$$[\underline{e}_r, \underline{e}_\theta] = -\frac{1}{r}\underline{e}_\theta$$

$(\underline{e}_r, \underline{e}_\theta)$ is not a coordinate basis

---

Proof

$$\underline{e}_r = \partial_r \, ; \quad \underline{e}_\theta = \frac{1}{r}\partial_\theta$$

$$[\underline{e}_r, \underline{e}_\theta]^i = \underline{e}_r^{\,j}\, \partial_j\, \underline{e}_\theta^{\,i} - \underline{e}^{\,j}_\theta\, \partial_j\, \underline{e}_r^{\,i}$$

$$= \partial_r \underline{e}_\theta^{\,i} - \partial_\theta \underline{e}_r^{\,i} = \partial_r \underline{e}_\theta^{\,i}$$

$$[\underline{e}_r, \underline{e}_\theta] = \partial_r \underline{e}_\theta = -\frac{1}{r^2}\partial_\theta = -\frac{1}{r}\underline{e}_\theta \quad \text{qed}$$

$$\partial_r\left(\frac{1}{r}\underline{e}_\theta\right) = \partial_r\left(\frac{1}{r}\partial_\theta\right) = -\frac{1}{r^2} = -\frac{1}{r}\underline{e}_r$$

Let's go back to the metric tensor $\underline{g}$. Given a vector $\underline{U}$ with unit norm, it is possible to introduce a <u>projection tensor</u> that splits any tensor of rank 2 in a direction parallel and perpendicular to $\underline{U}$

$$h_{\mu\nu} := g_{\mu\nu} + U_\mu U_\nu \odot \qquad \text{where} \quad U_\mu U^\mu = g_{\mu\nu} U^\mu U^\nu = -1$$

(unit norm)

Then it's not difficult to prove the following identities

(1) $h_{\mu\nu} U^\mu = 0$

(2) $h_\mu{}^\lambda h_{\lambda\nu} = h_{\mu\nu}$

(3) $h_\mu{}^\mu = g_{\mu\nu} h^{\mu\nu} = 3$

$\boxed{\text{of course also } \quad h^{\mu\nu} := g^{\mu\nu} + U^\mu U^\nu \quad \text{is a projection tensor}}$

Proofs

(1) $h_{\mu\nu} U^{\mu} = g_{\mu\nu} U^{\mu} + U_{\mu} U_{\nu} U^{\mu} = U_{\nu} + \underbrace{U_{\mu} U^{\mu}}_{-1} U_{\nu} = U_{\nu} - U_{\nu} = 0$

(2) $h_{\mu}{}^{\lambda} h_{\lambda\nu} = (g_{\mu}{}^{\lambda} + U_{\mu} U^{\lambda})(g_{\lambda\nu} + U_{\lambda} U_{\nu})$

$$= g_{\mu\nu} + \underbrace{U_{\mu} U_{\nu} + U_{\mu} U_{\nu} + U_{\mu} \underbrace{U^{\lambda} U_{\lambda}}_{-1} U_{\nu}}_{0} = g_{\mu\nu} + U_{\mu} U_{\nu} = h_{\mu\nu}$$

(3) $h_{\mu}{}^{\mu} = g_{\mu\nu} h^{\mu\nu} = g_{\mu\nu}(g^{\mu\nu} + U^{\mu} U^{\nu}) = \overbrace{4}^{\delta^{\mu}_{\mu} = 1+1+1+1} + U_{\nu} U^{\nu} = 4 - 1 = 3$ $\square$

Having now $\underline{h}$ and $\underline{U}$ we can split any tensor in a direction parallel to $\underline{U}$ and perpendicular to it.

Ex

$$V^\mu = \underset{\parallel \text{ to } \underline{U}}{A U^\mu} + \underset{\perp \text{ to } \underline{U}}{B^\mu} = A U^\mu + \underbrace{h^\mu{}_\nu V^\nu}_{\text{def. of } B^\mu}$$

then

$$U_\mu V^\mu = A U_\mu U^\mu + U_\mu B^\mu$$

$$= -A + U_\mu \left(h^\mu{}_\nu V^\nu \right) = -A + U_\mu \left(\delta^\mu{}_\nu + U^\mu U_\nu \right) V^\nu$$

$$= -A + \left(U_\nu + U_\mu U^\mu U_\nu \right) V^\nu = -A + \left(U_\nu - U_\nu \right) V^\nu = -A$$

In other words $-A$ is the projection of $\underline{V}$ along $\underline{U}$

and $\underline{B} \cdot \underline{U} = U_\mu B^\mu = 0$.

In a similar way it is possible to split also a generic tensor of rank 2 by applying the projection operator seperately on each component of the tensor

$$W_{\mu\nu} = A U_\mu U_\nu + B_\mu U_\nu + U_\mu C_\nu + Z_{\mu\nu}$$

where

$$A := W_{\mu\nu} U^\mu U^\nu \quad ; \quad B_\mu := -h^\alpha{}_\mu W_{\alpha\beta} U^\beta$$

$$C_\nu := -h^\alpha{}_\nu W_{\beta\alpha} U^\beta \quad ; \quad Z_{\mu\nu} := h^\alpha{}_\mu h^\beta{}_\nu W_{\alpha\beta}$$

We can also express $Z_{\mu\nu}$ in its symmetric and skew parts as

$$Z_{\mu\nu} = Z_{(\mu\nu)} + Z_{[\mu\nu]}$$

with $Z_{(\mu\nu)} = h^\alpha{}_{(\mu} h^\beta{}_{\nu)} W_{\alpha\beta} = \frac{1}{2}\left(h^\alpha{}_\mu h^\beta{}_\nu + h^\alpha{}_\nu h^\beta{}_\mu\right) W_{\alpha\beta}$

$$Z_{[\mu\nu]} = h^\alpha{}_{[\mu} h^\beta{}_{\nu]} W_{\alpha\beta} = \frac{1}{2}\left(h^\alpha{}_\mu h^\beta{}_\nu - h^\alpha{}_\nu h^\beta{}_\mu\right) W_{\alpha\beta}$$

$Z_{(\mu\nu)}$ can be further written as

$$Z_{(\mu\nu)} = W_{\langle\mu\nu\rangle} + \frac{1}{3} W_{\alpha\beta} h^{\alpha\beta} h_{\mu\nu}$$

the trace $(W_{\alpha\beta} h^{\alpha\beta})$ is removed

with

$$W_{\langle\mu\nu\rangle} := \underbrace{h^{\alpha}{}_{\mu} h_{\nu}{}^{\beta} W_{(\alpha\beta)}}_{Z_{(\mu\nu)}\ ①} - \frac{1}{3} W_{\alpha\beta} h^{\alpha\beta} h_{\mu\nu} :$$ <u>trace-free</u> part of $W_{\alpha\beta}$

Putting things together, we have

$$W_{\mu\nu} = A U_\mu U_\nu + B_\nu U_\mu + U_\mu C_\nu + Z_{(\mu\nu)} + Z_{[\mu\nu]}$$

$$= A_\mu U_\nu + B_\nu U_\mu + U_\mu C_\nu + \frac{1}{3} W_{\alpha\beta} h^{\alpha\beta} h_{\mu\nu} + W_{\langle\mu\nu\rangle} + h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} W_{[\alpha\beta]}.$$

① Note that $h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} W_{(\alpha\beta)} = h^{\alpha}{}_{(\mu} h^{\beta}{}_{\nu)} W_{\alpha\beta}$ (Exercise)

TENSOR CALCULUS

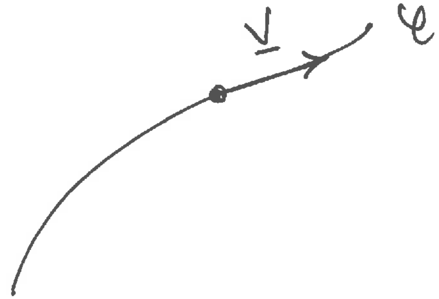
All what we have discussed so far is about tensor algebra (sum, products, contraction, etc...) of tensors. However, for more interesting is to look at how to study the variations (derivatives) of tensors. We will start with a simple extension of the normal partial derivative that is represented by the Lie derivative to move on to the concept of covariant derivative.

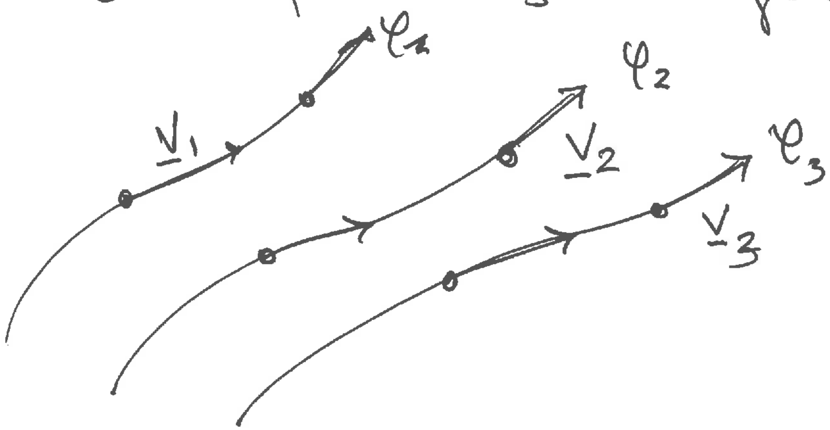
- LIE DERIVATIVE

In essence the Lie derivative expresses the derivative of a vector field relative to another vector field.

To see how this works let's recall that we can associate to a vector $\underline{V}$ a curve $\mathcal{C}$ to which it is tangent

If $\underline{V}$ is a vector field $\underline{V}(x^\mu)$ it will generate a family of curves, called $\mathcal{C}_{\underline{V}}$ the congruence of $\underline{V}$ and having the vector field as tangent.





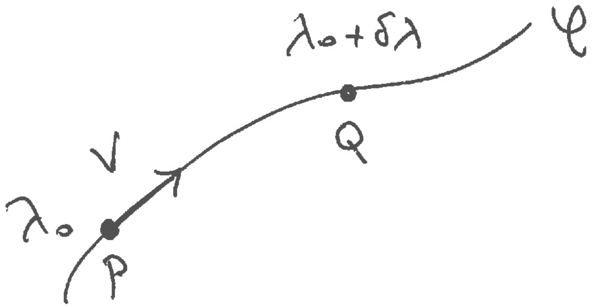
At each point P of coordinate $\{X^\mu\}$ in the congruence we have

$$V^\mu(x^\mu) := \frac{dx^\mu}{d\lambda}$$

where λ is a parameter along the curve.

The congruence $\mathcal{C}_{\underline{V}}$ provides a mapping of the manifold into itself. In other words, at any point P there is one and only one curve of the congruence $\mathcal{C}_{\underline{V}}$ passing though it.

Hence the point P can be mapped to a new point Q by dragging P along the curve $\mathcal{C}$



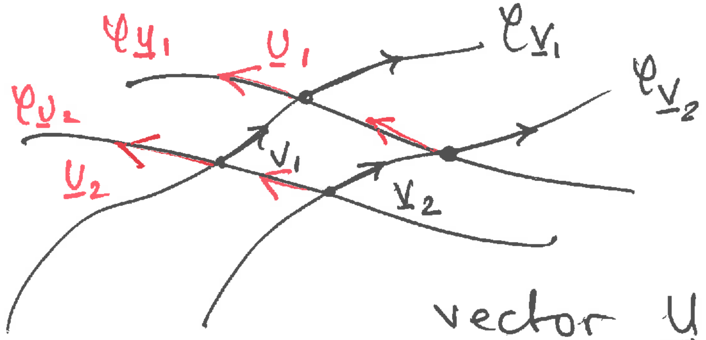
Mathematically we write that

$$Q = \phi_\lambda(P) \ ; \ \phi_\lambda \text{ mapping along } \mathcal{C}$$

where $P = \mathcal{C}(\lambda_0)$ and $Q = \mathcal{C}(\lambda_0 + \delta\lambda)$

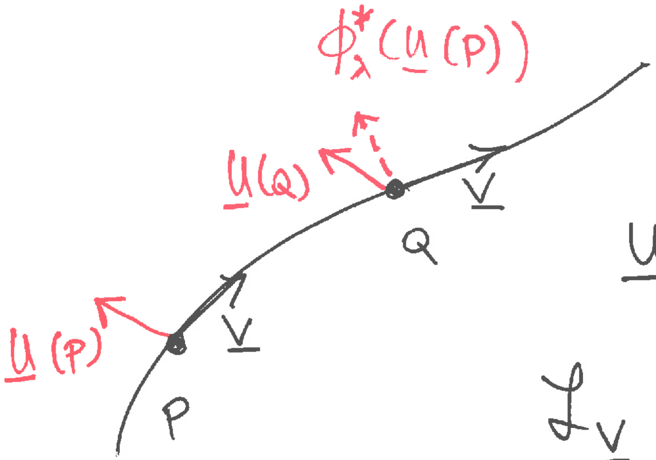
and $\mathcal{C}$ is the curve of the congruence passing through P and Q.

Consider now that the manifold also has another vector field $\underline{U}$ with associated congruence of curves $\mathcal{C}_{\underline{U}}$



What we can do now is to compare, ie measure the difference between the vector $\underline{U}(Q)$ and the vector $\underline{U}$ obtained after dragging the vector

$\underline{U}(P)$ from P to Q along the congruence $\mathcal{C}_{\underline{V}}$



In other words, the Lie derivative of $\underline{U}$ along $\underline{V}$ is given by

$$\mathcal{L}_{\underline{V}}\underline{U} := \lim_{\delta\lambda \to 0} \frac{\underline{U}(Q) - \phi^*_\lambda(\underline{U}(P))}{\delta\lambda} = \frac{\begin{pmatrix}\text{vector}\\ \text{field at}\\ Q\end{pmatrix} - \begin{pmatrix}\text{vector at}\\ P\text{ dragged}\\ \text{to } Q\end{pmatrix}}{\delta\lambda}$$

$$= \lim_{\delta\lambda \to 0} \frac{\underline{U}(\phi_\lambda(P)) - \phi^*_\lambda(\underline{U}(P))}{\delta\lambda}$$

$\underline{U}(\phi_\lambda(P))$ is the vector field $\underline{U}$ when dragged from P to Q along the congruence $\mathcal{C}_{\underline{V}}$, ie is the vector field at Q.

$\phi^*_\lambda(\underline{U}(P))$ is the vector obtained taking the tangent in P to the congruence $\mathcal{C}_{\underline{U}}$ and dragging it from P to Q

It is clear that this difference is zero if the two congruences $\mathcal{C}_{\underline{V}}$ and $\mathcal{C}_{\underline{Y}}$ are everywhere parallel,[⊙] but it is also clear that in general this difference can be non zero.
What does "shopping" mean mathematically? This is equivalent to a change of cordinates

$$P(x^\mu) \longrightarrow Q(\tilde{x}^\mu) \Longleftrightarrow \lambda_0 \longrightarrow \lambda_0 + \delta\lambda$$

$$x^\mu \longrightarrow \tilde{x}^\mu = x^\mu + \underbrace{\delta\lambda V^\mu}$$

differential increase of coordinates along $\underline{V}$: $\left(\delta\lambda \frac{dx^\mu}{d\lambda}\right)$

We can express the derivatives of $x^{\mu\prime}$ as

$$\partial_\nu x^\mu = \partial_\nu \left(x^\mu + \lambda V^\mu\right) = \delta^\mu{}_\nu + \delta\lambda\, \partial_\nu V^\mu$$

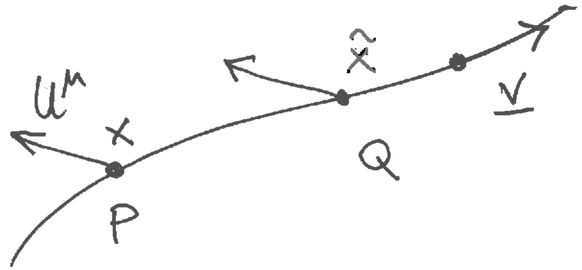
$$= \Lambda^\mu{}_\nu\, x^\nu$$

⊙ Indeed this can be taken as a definition of parallelism.

The changes of $\underline{U}$ when dragged from P to Q will be

$$\tilde{U}^\mu(\tilde{x}^\mu) = \Lambda^\mu{}_\nu U^\nu = \tilde{U}^\mu + \delta\lambda\, U^\nu(x)\,\partial_\nu V^\mu \qquad (\cdot)$$

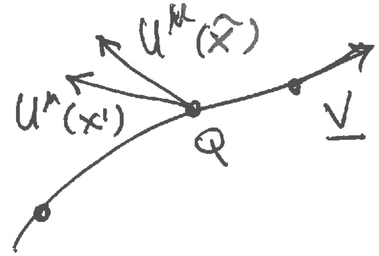
ie, the components in Q can be seen as those after a coordinate transformation



On the other hand $U^{\mu'}(x')$ can also be seen as a small variation of $U^{\mu'}(x)$ and thus expressed in terms of a Taylor expansion in term of the parameter λ, ie

$$U^\mu(\tilde{x}^\mu) = U^\mu(x^\mu) + \delta x^\nu\,\partial_\nu U^\mu + O(\delta\lambda^2) \qquad (\cdot\cdot)$$

$$= U^\mu(x^\mu) + \delta\lambda\, V^\nu\,\partial_\nu U^\mu$$



Combining (.) and (..) we have

$$\tilde{U}^\mu(x^\mu) = \tilde{U}^\mu + \delta\lambda\, U^\nu \partial_\nu V^\mu = U^\mu + \delta\lambda\, V^\nu \partial_\nu U^\mu$$

$$\Rightarrow \quad \tilde{U}^\mu - U^\mu = \delta\lambda \left(V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu \right)$$

so that

$$(\mathcal{L}_{\underline{V}}\, \underline{U})^\mu = \lim_{\delta\lambda \to 0} \frac{\tilde{U}^\mu - U^\mu}{\delta\lambda} = \boxed{V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu = (\mathcal{L}_{\underline{V}}\, \underline{U})^\mu}$$

In a similar way it is possible to derive the expression for the covariant components of the Lie derivative of $\underline{U}$ along $\underline{V}$

$$\boxed{(\mathcal{L}_{\underline{V}}\, \underline{U})_\mu = V^\nu \partial_\nu U^\mu + U^\nu \partial_\nu V^\mu}$$

Recap

- $\underline{e}_\alpha$ general (non-coordinate) basis $\quad \underline{e}_\alpha = \Lambda_\alpha{}^\mu \partial_\mu$

- $\underline{e}_\alpha$ coordinate basis if $\Lambda_\alpha{}^\mu = \delta_\alpha{}^\mu$

- $\underline{e}_\alpha$ orthonormal basis if $\underline{e}_\alpha \cdot \underline{e}_\beta = \eta_{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

- $\underline{Z} := [\underline{X}, \underline{Y}] := \underline{X}\,\underline{Y} - \underline{Y}\,\underline{X}$ (not scalar product!)

$$Z^\mu = [\underline{X}, \underline{Y}]^\mu = Y^\nu \partial_\nu X^\mu - X^\nu \partial_\nu Y^\mu$$

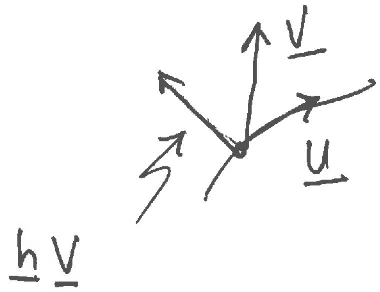
- $[\underline{e}_\mu, \underline{e}_\nu] = C^\lambda_{\mu\nu}\, \underline{e}_\lambda \qquad C^\lambda_{\mu\nu}$: structure coefficients

$$C^\lambda{}_{\mu\nu} = \left(e^\alpha_\mu \partial_\alpha e^\beta{}_\nu - e^\alpha_\nu \partial_\alpha e^\beta{}_\mu\right) e^\lambda_\beta$$

$C^{\lambda}_{\mu\nu} = 0 \iff \{\underline{e}_\mu\}$ is coordinate basis

Hence computing $C^{\lambda}_{\mu\nu}$ is a way to check whether basis is coordinate or not.

- Given $\underline{u}$: vector, $\underline{g}$: metric $\Longrightarrow$ $\underline{h}$ projector orthogonal to $\underline{u}$



$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

$$h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \quad ; \quad h^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu$$

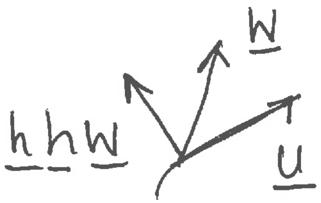
$$\underline{h}\,\underline{u} = 0$$

- $\underline{V}$ can be decomposed in part $//$ and $\perp$ to $\underline{u}$

$$V^\mu = A u^\mu + B^\mu = \underbrace{A u^\mu}_{//} + \underbrace{h^\mu{}_\nu V^\nu}_{\perp}$$

$A = -u_\mu V^\mu$

- Splitting is possible also for a rank 2 tensor $\underline{W}$



$$\underline{W} = \underline{W}\,\underline{u}\,\underline{u} + \underbrace{\underline{h}\,\underline{W}\,\underline{u} + \underline{u}\,\underline{W}\,\underline{h}}_{\text{different in general}} + \underline{h}\,\underline{h}\,\underline{W}$$

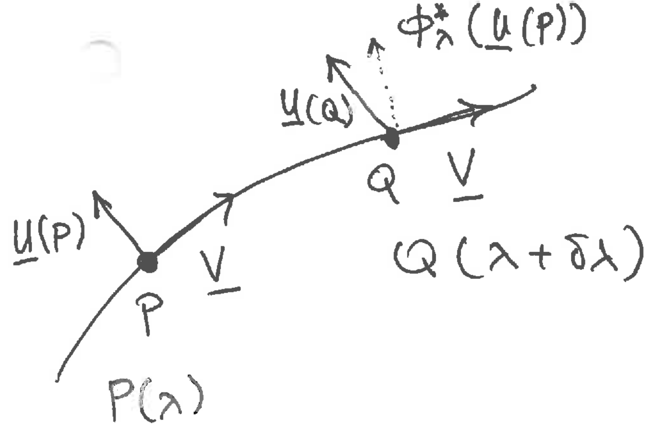
In components form

$$W_{\mu\nu} = A\, U_\mu U_\nu + B_\mu U_\nu + U_\mu C_\nu + Z_{\mu\nu}$$

$$A = W_{\mu\nu} U^\mu U^\nu \;; \quad B_\mu = -h^\alpha{}_\mu W_{\alpha\beta} U^\beta .$$

$$C_\nu = -h^\alpha{}_\nu W_{\alpha\beta} U^\beta \;; \quad Z_{\mu\nu} = h^\alpha{}_\mu h^\beta{}_\nu W_{\alpha\beta}$$

- Tensor calculus: Lie derivative, ie derivative of a vector field relative to another vector field. In a Lie derivative you need to specify which vector field you are deriving relative to.



$\underline{U}(P)$: vector field at P

$\underline{U}(Q)$: " " " Q ; $Q = \phi_\lambda(P)$

$\phi^*_\lambda(\underline{U}(P))$: " " at P dragged to Q

$$\mathcal{L}_{\underline{V}}\underline{U} = \lim_{\delta\lambda\to 0} \frac{\underline{U}(Q) - \phi^*_\lambda(\underline{U}(P))}{\delta\lambda} = \frac{\underline{U}(\phi_\lambda(P)) - \phi^*_\lambda(\underline{U}(P))}{\delta\lambda}$$

$$\underline{U}(Q) \quad \tilde{U}^\mu \overset{(\cdot)}{=} \Lambda^\mu{}_\nu U^\nu = \tilde{U}^\mu + \delta\lambda\, U^\nu \partial_\nu V^\mu$$

dragging is like coord. transf.

these are the same

$$\phi^*_\lambda(\underline{U}(P)) \longrightarrow \tilde{U}^\mu(x) = U^\mu + \delta\lambda\, V^\nu \partial_\nu U^\mu + O(\delta\lambda^2)$$

this can be seen as Taylor expansion

$(\cdot)$
$$\Lambda^{\mu'}{}_\nu = \frac{\partial x^{\mu'}}{\partial x^\nu} = \partial_\nu(x^\mu + \delta x^\mu) = \partial_\nu(x^\mu + \delta\lambda V^\mu) = \delta^\mu{}_\nu + \delta\lambda\,\partial_\nu V^\mu$$

$$\tilde{U}^\mu + \delta\lambda\, U^\nu \partial_\nu V^\mu = U^\mu + \delta\lambda\, V^\nu \partial_\nu U^\mu$$

$$\tilde{U}^\mu - U^\mu = \delta\lambda \left(V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu \right)$$

$$\left(\mathcal{L}_{\underline{V}}\, \underline{U} \right)^\mu = \lim_{\delta\lambda \to 0} \frac{\tilde{U}^\mu - U^\mu}{\delta\lambda} = V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu \quad \square$$

Similarly

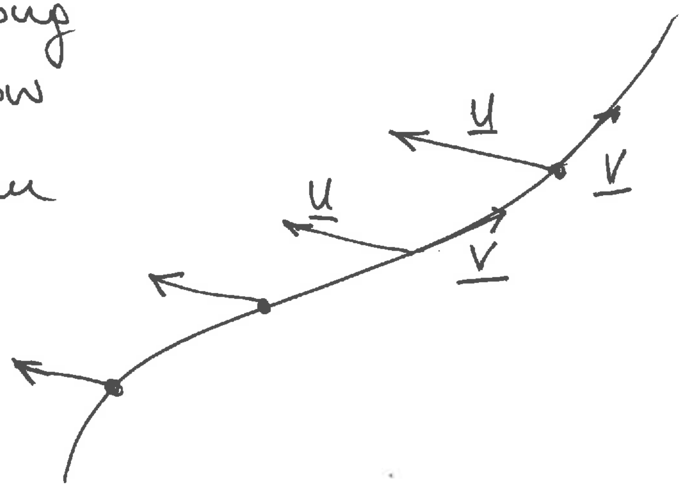
$$\left(\mathcal{L}_{\underline{V}}\, \underline{U} \right)_\mu = V^\nu \partial_\nu U_\mu + U^\nu \partial_\nu V_\mu$$

$$\left(\mathcal{L}_{\underline{V}}\,\underline{U}\right)^\mu = \underbrace{V^\nu \partial_\nu U^\mu}\ -\ \underbrace{U^\nu \partial_\nu V^\mu}$$

directional derivative, ie variation of $\underline{U}$ along the direction given $\underline{V}$

directional derivative of $\underline{V}$ along $\underline{U}$

In other words, when computing how the components of a vector change along a curve we should also consider how the tangent to the curve itself can change



Properties of the Lie derivative

1) $\mathcal{L}_{\underline{V}}\,\phi = V^\mu \partial_\mu \phi$ $\qquad\phi$: scalar function

2) $\mathcal{L}_{\phi\underline{V}}\,\underline{T} = \phi\,\mathcal{L}_{\underline{V}}\,\underline{T} - \underline{V}\,\mathcal{L}_{\underline{T}}\,\phi$

3) Linear operator : $\mathcal{L}_{\underline{V}}\left(a Y^{\alpha\nu} + b Z^{\beta\nu}\right) = a\,\mathcal{L}_{\underline{V}} Y^{\alpha\nu} + b\,\mathcal{L}_{\underline{V}} Z^{\beta\nu}$

4) Follows the Liebniz rules $\mathcal{L}_{\underline{V}}\left(Z^{\mu\nu} Y_{\alpha\beta}\right) = \left(\mathcal{L}_{\underline{V}} Z^{\mu\nu}\right) Y_{\alpha\beta} +$

$$\left(\mathcal{L}_{\underline{V}} Y_{\alpha\beta}\right) Z^{\mu\nu}$$

5) It's "type preserving"

$$\mathcal{L}_{\underline{V}}\left[\begin{pmatrix} m \\ n \end{pmatrix}\text{form}\right] = \begin{pmatrix} m \\ n \end{pmatrix}\text{form}$$

6) commutes with contraction $\delta^\mu_\nu \mathcal{L}_{\underline{V}}\, T^\nu{}_\mu = \mathcal{L}_{\underline{V}}\, T^\mu{}_\mu$

7) when acting on higher than rank-1 tensor it gains a positive/negative sign for each covariant/contravariant index

$$\mathcal{L}_{\underline{V}} T^{\alpha}{}_{\beta} = V^{\mu} \partial_{\mu} T^{\alpha}{}_{\beta} - T^{\mu}{}_{\beta} \partial_{\mu} V^{\alpha} + T^{\alpha\mu} \partial_{\mu} V_{\beta}$$

$$\mathcal{L}_{\underline{V}} Z_{\alpha\beta} = V^{\mu} \partial_{\mu} Z_{\alpha\beta} + Z_{\alpha}{}^{\mu} \partial_{\mu} V_{\beta} + Z^{\mu}{}_{\beta} \partial_{\mu} V_{\alpha}$$

8) Recalling definition of commutator of two vectors

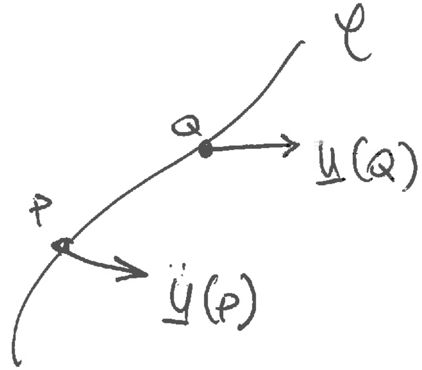
$$\mathcal{L}_{\underline{V}} \underline{U} = [\underline{V}, \underline{U}] = \underline{V}\,\underline{U} - \underline{U}\,\underline{V}$$

$$\mathcal{L}_{\underline{V}} U^{\mu} = V^{\alpha} \partial_{\alpha} U^{\mu} - U^{\mu} \partial_{\mu} V^{\alpha} = [\underline{V}, \underline{U}]^{\mu}$$

At this point one may wonder why use a more complicated expression for the derivative when we could use simply the partial derivative.

What's wrong with a simpler definition of

$$\left(\begin{array}{l} \text{derivative} \\ \text{of } \underline{U} \text{ along } \mathcal{C} \end{array} \right) := \lim_{\delta\lambda \to 0} \frac{(U^\mu)_Q - (U^\mu)_P}{\delta\lambda}$$



It's not difficult to realize that such definition would not be satisfactory; let's consider how this definition would transform after a coordinate transformation

$$(U^{\mu'})_P = (\Lambda^{\mu'}{}_{\mu})_P (U^\mu)_P \quad ; \quad (U^{\mu'})_Q = (\Lambda^{\mu'}{}_{\mu})_Q (U^\mu)_Q$$

$$(\text{derivative}) = \lim_{\delta\lambda \to 0} \frac{(\Lambda^{\mu'}{}_{\mu})_Q \, (U^{\mu})_Q - (\Lambda^{\mu'}{}_{\mu})_P \, (U^{\mu})_P}{\delta\lambda}$$

these are the transformation matrices at Q and P but there is no information on how these matrices change between P and Q.

Another way to realize this is to study how a simple partial derivative changes after coordinate transformation

Consider $\partial_\nu V^\mu = T_\nu{}^\mu$: 2 indices object

$$T_\nu{}^\mu \rightarrow T_{\nu'}{}^{\mu'} = \partial_{\nu'} V^{\mu'} = \partial_{\nu'} \left(\Lambda^{\mu'}{}_{\mu} V^\mu \right) =$$

$$= \frac{\partial}{\partial x^{\nu'}} \left(\frac{\partial x^{\mu'}}{\partial x^{\mu}} V^\mu \right) = \frac{\partial^2 x^{\mu'}}{\partial x^{\nu'} \partial x^{\mu}} V^\mu + \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial V^\mu}{\partial x^{\nu'}}$$

$$= \frac{\partial^2 x^{\mu'}}{\partial x^{\nu'} \partial x^{\mu}} V^\mu + \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial V^\mu}{\partial x^{\nu}}$$

$$= \frac{\partial^2 x^{\mu'}}{\partial x^{\nu'} \partial x^{\mu}} V^{\mu} + \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu}{}_{\nu'} \underbrace{\partial_{\nu} V^{\mu}}_{T_{\nu}{}^{\mu}}$$

In other words

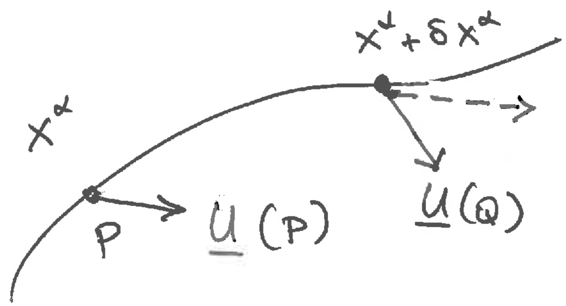
$$T_{\nu'}{}^{\mu'} = \underbrace{\Lambda^{\mu'}{}_{\mu} \Lambda^{\nu}{}_{\nu'} T_{\nu}{}^{\mu}}_{\text{this part transforms like a } \binom{1}{1} \text{ tensor}} + \underbrace{\frac{\partial^2 x^{\mu'}}{\partial x^{\nu'} \partial x^{\mu}} V^{\mu}}_{\text{this part is extra and nonzero in general}}$$

In other words partial derivatives of vectors are not tensors.

We need a definition of derivative that is covariant, ie that doesn't depend on the coordinates.

There are several ways of deriving such a derivative

Covariant derivative



$U^\mu(x^\alpha)$

$U^\mu(x^\alpha + \delta x^\alpha)$

Let δU^μ be the difference between $\underline{U}(Q)$ and $\underline{U}(P)$, ie

$$\delta U^\mu = U^\mu(x^\alpha + \delta x^\alpha) - U^\mu(\delta x^\alpha) = \delta x^\alpha \partial_\alpha V^\mu + O((\delta x^\alpha)^2)$$

$$= \underline{U}(Q) - \underline{U}(P)$$

Taylor expansion

Let instead $\overline{\delta U^\mu}$ be the difference between the vector $\underline{U}$ in Q ($\underline{U}(Q)$) and the vector $\underline{U}$ in Q that is parallel to the vector $\underline{U}$ in P

$$\overline{\delta \underline{U}} = \underline{U}(Q) - \underline{U}(Q)\Big|_{/\!/ \underline{U}(P)}$$

I can define the covariant derivative as the difference between these two vectors

$$\begin{pmatrix}\text{covariant}\\ \text{derivative}\end{pmatrix}^\mu := \lim_{\delta x^\alpha \to 0} \frac{\delta U^\mu - \overline{\delta U}^\mu}{\delta x^\alpha} = \lim_{\delta x^\alpha \to 0} \frac{1}{\delta x^\alpha}\left[\delta x^\alpha \partial_\alpha U^\mu - \overline{\delta U}^\mu\right]$$

$$= \nabla_\alpha U^\mu$$

I still don't know what is $\overline{\delta U}^\mu$ but I know it must be linear in δx^α, eg

$$\overline{\delta U}^\mu = - \Gamma^\mu_{\alpha\beta}\, U^\beta\, \delta x^\alpha$$

Putting things together: $\boxed{\nabla_\alpha U^\mu = \partial_\alpha U^\mu + \Gamma^\mu_{\alpha\beta} U^\beta}$ $\quad (*)$

Expression $(*)$ measures therefore the difference between a vector and its expression when parallel transported to a new position.

There is also a different way of deriving the expression of the covariant derivative and that is not based on geometry. In a general basis $\underline{U} = U^\mu \underline{e}_\mu$

$$\partial_\nu \underline{U} = \partial_\nu (U^\mu \underline{e}_\mu) = (\partial_\nu U^\mu) \underline{e}_\mu + U^\mu \partial_\nu \underline{e}_\mu$$

we can express $\partial_\nu \underline{e}_\mu$ as another vector with components in the same basis

$$\partial_\nu \underline{e}_\mu = \Gamma^\lambda_{\mu\nu} \underline{e}_\lambda$$

this is the change of the coordinates themselves between P and Q

so that

$$\partial_\nu \underline{U} = (\partial_\nu U^\mu) \underline{e}_\mu + \Gamma^\lambda_{\mu\nu} \underline{e}_\lambda U^\mu =$$

$$= (\partial_\nu U^\mu) \underline{e}_\mu + \Gamma^\mu_{\lambda\nu} \underline{e}_\mu U^\lambda =$$

$$= (\partial_\nu U^\mu + \Gamma^\mu_{\lambda\nu} U^\lambda) \underline{e}_\mu =: (\nabla_\nu U^\mu) \underline{e}_\mu \Rightarrow$$

$$\nabla_\nu U^\mu = \partial_\nu U^\mu + \Gamma^\mu_{\lambda\nu} U^\lambda$$ which of course coincides with expression (*)

NOTES

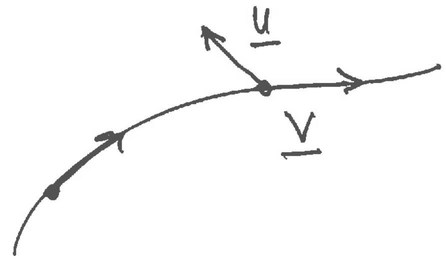
- $\Gamma^\alpha_{\beta\gamma}$: Chistoffel's symbols or affine coefficients
- $\Gamma^\alpha_{\beta\gamma}$ are not tensors (cf coordinate transform.; Exercise)
- the covariant derivative of a vector can be seen as composed of two parts: one expressing how the components change from place to place (partial derivative) and one part expressing how the coordinates themselves change from position to position (affine coefficients). This last part can also be zero
- Alternative notation (comma / semicolon) $\left.\begin{array}{l}\nabla_\nu U^\mu = U^\mu{}_{;\nu} \\ \partial_\nu U^\mu = U^\mu{}_{,\nu}\end{array}\right\} \Longrightarrow$

Recap

tensor

- Lie derivative : change of a vector field relative to another vector field.

$$(\mathcal{L}_{\underline{V}}\,\underline{U})^\nu = V^\mu \partial_\mu U^\nu - U^\mu \partial_\mu V^\nu$$



$$(\mathcal{L}_{\underline{V}}\,\underline{U})_\nu = V^\mu \partial_\mu U_\nu + U^\mu \partial_\nu V^\nu$$

change of vector $\underline{V}$ itself

- Partial derivatives of vector components are not sufficient. as they miss a term to behave as tensor under coordinate transformation

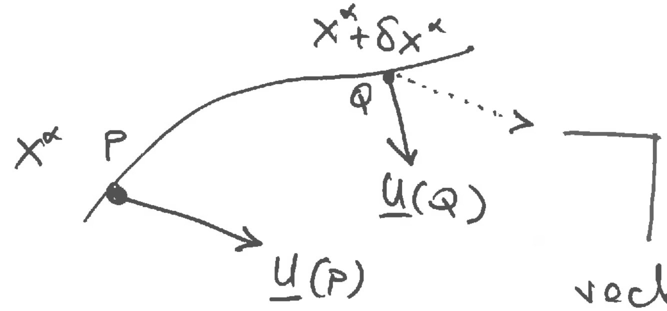
$$\partial_\nu V^\mu \rightarrow \partial_{\nu'} V^{\mu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu}{}_{\nu'} \partial_\nu V^\mu + \frac{\partial^2 x^{\mu'}}{\partial x^{\nu'} \partial x^\mu} V^\mu$$

Careful $\left(\dfrac{\partial^2 x^{\mu'}}{\partial x^{\nu'}\partial x^{\mu}}\right)V^{\mu} \neq 0$

$$\frac{\partial x^{\mu'}}{\partial x^{\nu'}} = \delta^{\mu'}{}_{\nu'} \;; \qquad \frac{\partial}{\partial x^{\mu}}\left(\delta^{\mu'}{}_{\nu'}\right)V^{\mu} = 0 \cdot V^{\mu} = 0$$

true only in coordinate basis but non true in non-coordinate basis (eg (r, θ)).

- covariant derivative



vector $\underline{U}(P)$ dragged over to Q keeping it parallel to vector $\underline{U}(P)$

$$\overline{\delta \underline{U}} = \underline{U}(Q) - \underline{U}(Q)\Big|_{/\!/ \underline{U}(P)}$$

$$\delta \underline{U} = \underline{U}(Q) - \underline{U}(P)$$

$$\left(\nabla_{\alpha}\underline{U}\right)^{\mu} = \lim_{\delta x^{\alpha} \to 0} \frac{\delta U^{\mu} - \overline{\delta U}^{\mu}}{\delta x^{\alpha}} = \nabla_{\alpha} U^{\mu}$$

• To find explicit expression consider

$$\partial_\nu (\underline{u}) = \partial_\nu (u^\mu \underline{e}_\mu) = (\partial_\nu u^\mu)\, \underline{e}_\mu + u^\mu \underbrace{(\partial_\nu \underline{e}_\mu)}_{\Gamma^\lambda_{\nu\mu}\, \underline{e}_\lambda}$$

$$= (\partial_\nu u^\mu + \Gamma^\mu_{\nu\alpha} u^\alpha)\, \underline{e}_\mu$$

$$= (\nabla_\nu u^\mu)\, \underline{e}_\mu$$

$$\Rightarrow \quad \nabla_\nu u^\mu = \partial_\nu u^\mu + \Gamma^\mu_{\nu\alpha} u^\alpha$$

ordinary partial derivative

information on how coordinates change from P to Q

- $\Gamma^{\alpha}_{\beta\mu}$: Christoffel symbols

- $\Gamma^{\alpha}_{\beta\mu}$: not tensors

- Alternative notation : colon, semi colon

$$\partial \longleftrightarrow , \qquad \nabla \longleftrightarrow ;$$

$$\nabla_\nu U^\mu = U^\mu{}_{;\nu} \qquad \partial_\nu U^\mu = U^\mu{}_{,\nu}$$

$$\boxed{U^{\mu}{}_{;\nu} = U^{\mu}{}_{,\nu} + \Gamma^{\mu}{}_{\lambda\nu} U^{\lambda}}$$ □

Properties of the covariant derivative

1) $\nabla_\alpha \phi = \partial_\alpha \phi = \phi_{,\alpha}$

2) $\nabla_\alpha V_\beta = \partial_\alpha V_\beta - \Gamma_{\alpha}{}^{\mu}{}_{\beta} V_\mu$

3) $\Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta} = T^{\alpha}{}_{\beta\gamma}$: torsion tensor

so, while the christoffel's symbols are not tensors their difference is a tensor and the corresponding tensor is called torsion. and is obviously antisymmetric.
Einstein's theory assumes torsion-free coordinates but extensions are possible in which $T \neq 0$ (Einstein-Cartan)

4) $\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta}$ in torsion free spacetimes as we will

consider hereafter.

5) $$\mathcal{L}_{\underline{V}} U^\mu = V^\alpha \partial_\alpha U^\mu - U^\alpha \partial_\alpha V^\mu = V^\alpha \left[\partial_\alpha U^\mu + \Gamma^\mu_{\alpha\beta} U^\beta\right] - U^\alpha \left[\partial_\alpha V^\mu + \Gamma^\mu_{\alpha\beta} V^\beta\right]$$

these are the same

$$= V^\alpha \nabla_\alpha U^\mu - U^\alpha \nabla_\alpha V^\mu$$

In other words

$$\mathcal{L}_{\underline{V}} \underline{U} = \underline{V}(\nabla \underline{U}) - \underline{U}(\nabla \underline{V}) = \nabla_{\underline{V}} \underline{U} - \nabla_{\underline{U}} \underline{V}$$

(Lie deriv. of $\underline{U}$ along $\underline{V}$) = (covar. derivative of $\underline{U}$ along $\underline{V}$) − (covariant derivative of $\underline{V}$ along $\underline{U}$)

6) Covariant derivative of covector

$$\boxed{\nabla_\nu U_\mu = \partial_\nu U_\mu - \Gamma^\alpha_{\mu\nu} U_\alpha}$$

Proof

Consider the scalar product $U_\mu V^\mu$

$$\nabla_\nu (U_\mu V^\mu) = (\nabla_\nu U_\mu) V^\mu + U_\mu \nabla_\nu V^\mu$$

$$= \partial_\nu (\underbrace{U_\mu V^\mu}_{\text{scalar}}) = (\nabla_\nu U_\mu) V^\mu + U_\mu (\partial_\nu V^\mu + \Gamma^\mu_{\nu\alpha} V^\alpha) \Longrightarrow$$

$$(\nabla_\nu U_\mu) V^\mu = \partial_\nu (U_\mu V^\mu) - U_\mu (\partial_\nu V^\mu + \Gamma^\mu_{\nu\alpha} V^\alpha)$$

This tensor equation is true for any vector V^μ and thus also for $V^\mu \longrightarrow \delta^\mu{}_\beta$

$$(\nabla_\nu U_\mu) \delta^\mu{}_\beta = \partial_\nu (U_\mu \delta^\mu{}_\beta) - U_\mu (\cancel{\partial_\nu \delta^\mu{}_\beta} + \Gamma^\mu_{\nu\alpha} \delta^\alpha{}_\beta)$$

$$= \partial_\nu U_\beta - U_\mu (\Gamma^\mu_{\nu\beta}) \Longleftrightarrow$$

$$\nabla_\nu U_\beta = \partial_\nu U_\beta - \Gamma^\mu_{\nu\beta} U_\mu \qquad \text{qed}$$

7) $\boxed{\underline{\nabla}\, \underset{\sim}{g} = 0} \iff \nabla_\mu g_{\alpha\beta} = 0$

covariant derivative of the metric is zero (ie metric is invariant to covariant derivative)

Proof

$$\nabla_\alpha A_\mu = \nabla_\alpha (g_{\mu\nu} A^\nu) = (\nabla_\alpha g_{\mu\nu}) A^\nu + g_{\mu\nu} \nabla_\alpha A^\nu$$

Covariant derivative is a tensor, ie $\nabla_\alpha A^\nu = T_\alpha{}^\nu$

and $g_{\mu\nu} T_\alpha{}^\nu = T_{\alpha\mu} = \nabla_\alpha A_\mu \Rightarrow \nabla_\alpha g_{\mu\nu} = 0$ qed

How do I calculate the Christoffel symbols?

We need to use repeatedly the invariance of the metric tensor.

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} - \Gamma^\alpha_{\lambda\nu} g_{\mu\alpha} = 0 \qquad (1)$$

$$\nabla_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} - \Gamma^\alpha_{\mu\lambda} g_{\nu\alpha} = 0 \qquad (2)$$

$$\nabla_\nu g_{\lambda\mu} = \partial_\nu g_{\lambda\mu} - \Gamma^\alpha_{\nu\lambda} g_{\alpha\mu} - \Gamma^\alpha_{\nu\mu} g_{\lambda\alpha} = 0 \qquad (3)$$

$(2) + (3) - (1) = 0 \iff$

$$\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} - 2\Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} \implies$$

$$g_{\alpha\lambda}\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right)$$

Multiply by $g^{\beta\lambda}$ $\qquad (g^{\beta\lambda} g_{\alpha\lambda} = \delta^\beta{}_\alpha)$

$$\boxed{\Gamma^\beta{}_{\mu\nu} = \frac{1}{2} g^{\beta\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right)}$$

It is also clear that a in Cartesian coordinates

$$g_{ij} = \delta_{ij}\, dx^i dx^j$$

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij} \right)$$

$$= \frac{1}{2} \delta^{kl} \left(\partial_i \delta_{jl} + \partial_j \delta_{li} - \partial_l \delta_{ij} \right) = 0$$

The Christoffel's symbols are zero in Cartesian coordinates

Another example: $ds^2 = dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$

We can employ the identity proven in the exercises, ie

$$\nabla_\mu U^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, U^\mu \right) \iff \nabla_i U^i = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g}\, U^i \right) = \vec{\nabla} \cdot \vec{U}$$

in a coordinate basis.

to compute the covariant divergence in spherical polar coordinates. We first recall that $\{e_r, e_\theta, e_\phi\}$ is not a coordinate basis and so we need to correct for this, ie

$$\nabla_i u^i = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g}\, u^i\right) = \frac{1}{\sqrt{g}} \left[\partial_r \left(\sqrt{g}\, u^r\right) + \frac{1}{r} \partial_\theta \left(\sqrt{g}\, u^\theta\right) + \frac{1}{r\sin\theta} \partial_\phi \left(\sqrt{g}\, u^\phi\right) \right]$$

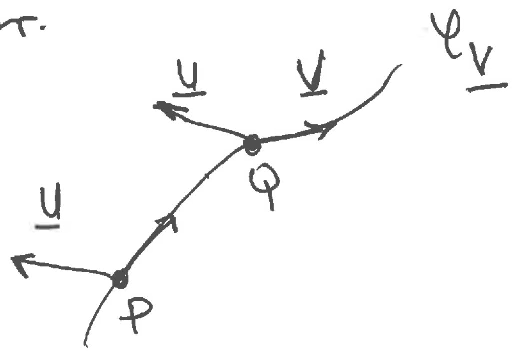
$$= \frac{1}{r^2 \sin\theta} \left[\partial_r \left(r^2 \sin\theta\, u^r\right) + \frac{1}{r} \partial_\theta \left(r^2 \sin\theta\, u^\theta\right) + \frac{1}{r\sin\theta} \partial_\phi \left(r^2 \sin\theta\, u^\phi\right) \right]$$

$g = r^4 \sin^2\theta$

$$= \frac{1}{r^2} \partial_r \left(r^2 u^r\right) + \frac{1}{r\sin\theta} \partial_\theta \left(\sin\theta\, u^\theta\right) + \frac{1}{r\sin\theta} \partial_\phi u^\phi$$

which is the well-known expression for the divergence of the three vector $\vec{u}$ in spherical coordinates.

Let's go back to a concept we have touched upon when discussing the covariant derivative but that we have not looked at very carefully. We have seen that it is possible and useful to think about the possibility of dragging a vector along a curve without changing the direction in which it is pointing. This operation is called "parallel transport" and is actually very simple to define using the tools we have derived so far.



From a geometrical point of view we want to make sure that the difference of the vector $\underline{u}$ at P and the one dragged from P to Q is zero

If $\underline{V}$ is the tangent vector of the curve $\mathscr{C} \in \mathscr{C}_{\underline{V}}$, then the vector field $\underline{U}$ is parallely transported along $\mathscr{C}$ iff

$$(1) \quad \boxed{\nabla_{\underline{V}} \underline{U} = 0} \iff \left(\begin{array}{l}\text{the covariant derivative of } \underline{U} \text{ along} \\ \text{the direction given by } \underline{V} \text{ is zero}\end{array}\right)$$

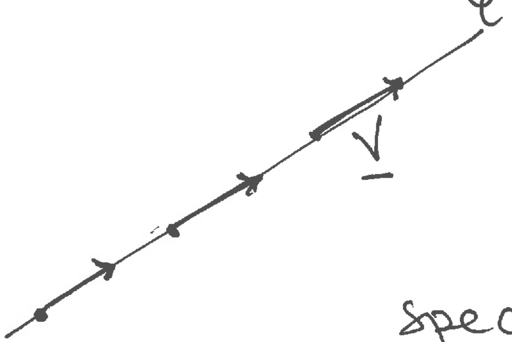
$$\Updownarrow$$

$$V^\mu \nabla_\mu U^\nu = 0 \iff V^\mu \partial_\mu U^\nu + \Gamma^\nu_{\mu\alpha} U^\alpha V^\mu = 0 \quad \square$$

We can also think that the operation of parallel transport can be used to single out a very special curve, namely, the curve that parallel transports its tangent vector

$$\mathscr{C}: \quad \nabla_{\underline{V}} \underline{V} = 0 \iff \boxed{V^\mu \nabla_\mu V^\nu = 0} \quad (2)$$

What is such a curve on a plane? This is a straight line!



Hence a curve that parallel transports its tangent vector can be seen as the generalization of a straight line also in spaces that are not restricted to a plane, ie curved spaces. We will look at curvature also later on, but for the time being let's write explicitely the expression of such curve (2)

By definition $V^\mu := \frac{dx^\mu}{d\lambda}$ $\Longrightarrow$ (2) can be written as

$$\nabla_{\underline{V}} \underline{V} = 0 \iff V^\mu \partial_\mu (V^\nu) + \Gamma^\nu_{\alpha\mu} V^\alpha V^\mu = 0 \iff$$

$$\frac{dx^\mu}{d\lambda} \frac{d}{dx^\mu} \left(\frac{dx^\nu}{d\lambda} \right) + \Gamma^\nu_{\alpha\mu} \frac{dx^\alpha}{d\lambda} \frac{dx^\mu}{d\lambda} = 0$$

$$\boxed{\frac{d^2 x^\nu}{d\lambda^2} + \Gamma^\nu_{\alpha\mu} \frac{dx^\alpha}{d\lambda} \frac{dx^\mu}{d\lambda} = 0} \quad (3) \qquad \underline{\text{geodesic curve}}$$

In other words the solution of the ordinary differential equation (3) is a geodesic and coincides with a straight line if all the Christoffel symbols are zero.. In this case, infact,

$$\frac{d^2 x^\nu}{d\lambda^2} = 0 \implies \frac{dx^\nu}{d\lambda} = k \implies \boxed{x^\nu = k\lambda}.$$

It is also not difficult to show that the geodesic curve is a curve of extremal proper length.

let $\quad \delta := \int_A^B 2\mathcal{L}\, d\lambda = \int_A^B ds = \int_A^B \sqrt{|g_{\mu\nu}\, dx^\mu dx^\nu|} = \int_A^B \sqrt{|g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu|}\, d\lambda$

where $\quad \dot{x}^\mu := \dfrac{dx^\mu}{d\lambda}$

$\boxed{\text{Recap}}$

- Properties of covariant derivative

$$\nabla_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^\beta_{\alpha\mu} V^\mu \qquad V^\beta{}_{;\alpha} = V^\beta{}_{,\alpha} + \Gamma^\beta_{\alpha\mu} V^\mu$$

$$\nabla_\alpha V_\beta = \partial_\alpha V_\beta - \Gamma_{\alpha\beta}^{\mu} V_\mu \qquad V_{\beta;\alpha} = V_{\beta,\alpha} - \Gamma_{\alpha\beta}^{\mu} V_\mu$$

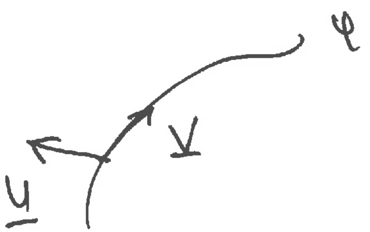
- $\mathcal{L}_{\underline{v}}\, \underline{u} = \nabla_{\underline{v}}\, \underline{u} - \nabla_{\underline{u}}\, \underline{v} = -[\underline{u}, \underline{v}]$

- $\nabla_\mu g_{\alpha\beta} = 0 = \nabla_\mu g^{\alpha\beta}$

- $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} \left(\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\gamma\mu} - \partial_\mu g_{\beta\gamma} \right)$

- $$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, V^\mu\right)$$

- Parallel transport : drag a vector along a curve without changing the direction it is pointing into.



$$\nabla_{\underline{V}}\,\underline{u} = 0 \iff \left(\begin{array}{l}\text{dragging of } \underline{u} \text{ along}\\ \text{the curve having } \underline{V} \text{ as}\\ \text{tangent vector is zero}\end{array}\right)$$

$$\iff (\nabla_{\underline{V}}\,\underline{u})^\mu = 0 \iff V^\nu \nabla_\nu u^\mu = 0$$

$$V^\nu\left(\partial_\nu u^\mu + \Gamma^\mu_{\nu\alpha} u^\alpha\right) = 0$$

- A curve is selected by its tangent vector field. Hence, there is a special curve that is selected by the parallel transport of its tangent vector

$$\nabla_{\underline{V}} \underline{V} = 0 \iff V^\nu \left(\partial_\nu V^\mu + \Gamma^\mu_{\nu\alpha} V^\alpha \right) = 0$$

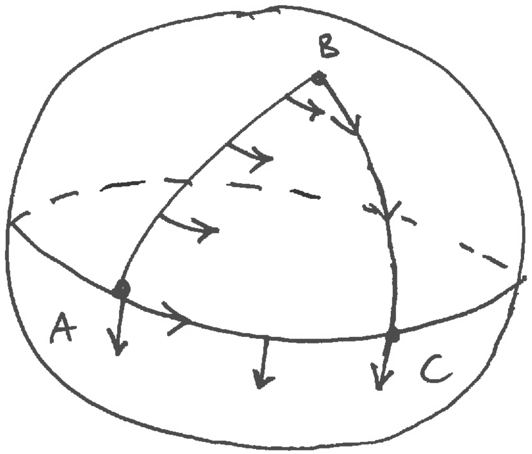
$$V^\mu = \frac{dx^\mu}{d\lambda} \qquad \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\alpha} \frac{dx^\nu}{d\lambda} \frac{dx^\alpha}{d\lambda} = 0$$ geodesic curve

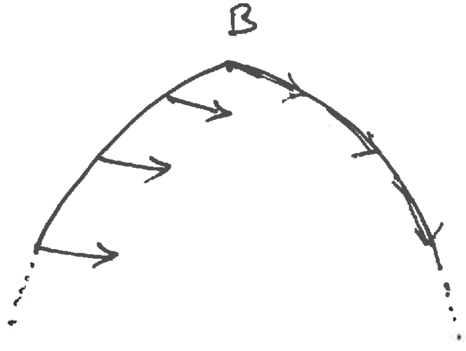
- On a flat surface all Γ's are zero $\Rightarrow$ geodesic curve is simply $x^\mu = k\lambda$, ie a straight line. On a non-flat surface it will be the equivalent of a straight line.

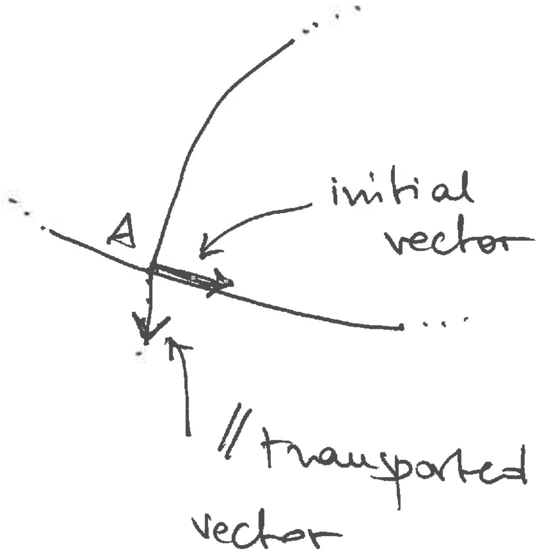
- In all cases (flat/non-flat) the geodesic curve is the curve of minimal length

$$\delta \int_A^B ds = 0 \iff \ddot{x}^\mu + \Gamma^\mu_{\alpha\nu} \dot{x}^\nu \dot{x}^\alpha = 0 \qquad \dot{x}^\mu = dx^\mu / d\lambda$$

However, it is also clear that there are easy counterexamples.
Let's think of a parallel transport on a 2-sphere







Clearly the two vectors are different: 2-sphere is not "flat"

What is clear is that we could use the parallel transport of a vector to measure the "curvature" of the 2-sphere. The essential operation in this case is that of second-order covariant derivative.

We know partial derivatives commute but this is not necessarily the case for covariant derivatives, ie

$$\partial_\alpha \partial_\beta V^\mu = \partial_\beta \partial_\alpha V^\mu \qquad \text{always}$$

$$\nabla_\alpha \nabla_\beta V^\mu \neq \nabla_\beta \nabla_\alpha V^\mu \qquad \text{in general.}$$

Let's compute such double derivative and in particular its difference, ie $2\nabla_{[\alpha}\nabla_{\beta]} V^\mu = \nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu$

$$= -\left(\nabla_\beta \nabla_\alpha V^\mu - \nabla_\alpha \nabla_\beta V^\mu\right)$$

$$\nabla_\beta \nabla_\alpha V^\mu =$$

$$= \nabla_\beta \left(\partial_\alpha V^\mu + \Gamma^\mu_{\alpha\nu} V^\nu\right) = \nabla_\beta T^\mu{}_\alpha$$

$$= \partial_\beta T^\mu{}_\alpha + \Gamma^\mu_{\beta\nu} T^\nu{}_\alpha - \Gamma^\nu_{\beta\alpha} T^\mu{}_\nu$$

$$= \partial_\beta \left(\partial_\alpha V^\mu + \Gamma^\mu_{\alpha\nu} V^\nu \right) + \Gamma^\mu_{\beta\nu} \left(\partial_\alpha V^\nu + \Gamma^\nu_{\alpha\delta} V^\delta \right) - \Gamma^\nu_{\beta\alpha} \left(\partial_\nu V^\mu + \Gamma^\mu_{\nu\delta} V^\delta \right)$$

$$= \partial_\beta \partial_\alpha V^\mu + (\partial_\beta \Gamma^\mu_{\alpha\nu}) V^\nu + \Gamma^\mu_{\alpha\nu} \partial_\beta V^\nu + \Gamma^\mu_{\beta\nu} \partial_\alpha V^\nu + \Gamma^\mu_{\beta\nu} \Gamma^\nu_{\alpha\delta} V^\delta$$

$$- \Gamma^\nu_{\beta\alpha} \partial_\nu V^\mu - \Gamma^\nu_{\beta\alpha} \Gamma^\mu_{\nu\delta} V^\delta .$$

Similarly

$$\nabla_\alpha \nabla_\beta V^\mu = \partial_\alpha \partial_\beta V^\mu + (\partial_\alpha \Gamma^\mu_{\beta\nu}) V^\nu + \Gamma^\mu_{\beta\nu} \partial_\alpha V^\nu + \Gamma^\mu_{\alpha\nu} \partial_\beta V^\nu + \Gamma^\mu_{\alpha\nu} \Gamma^\nu_{\beta\delta} V^\delta$$

$$- \Gamma^\nu_{\alpha\beta} \partial_\nu V^\mu - \Gamma^\nu_{\alpha\beta} \Gamma^\mu_{\nu\delta} V^\delta$$

As a result, the difference is

$$2 \nabla_{[\beta} \nabla_{\alpha]} V^\mu = \nabla_\beta \nabla_\alpha V^\mu - \nabla_\alpha \nabla_\beta V^\mu$$

$$= \partial_\beta \cancel{\partial_\alpha} V^\mu + (\partial_\beta \Gamma^\mu_{\alpha\nu}) V^\nu + \cancel{\Gamma^\mu_{\alpha\nu} \partial_\beta V^\nu} + \Gamma^\mu_{\beta\nu} \partial_\alpha V^\nu$$

$$+\Gamma^\mu_{\beta\nu}\Gamma^\nu_{\alpha\delta}V^\delta - \overline{\Gamma^\nu_{\beta\alpha}\partial_\nu V^\mu} - \cancel{\Gamma^\nu_{\beta\alpha}\Gamma^\mu_{\nu\delta}V^\delta} - \cancel{\partial_\alpha\partial_\beta V^\mu} - (\partial_\alpha\Gamma^\mu_{\beta\nu})V^\nu$$

$$-\Gamma^\mu_{\beta\nu}\partial_\alpha V^\nu - \Gamma^\mu_{\alpha\nu}\partial_\beta V^\nu - \Gamma^\mu_{\alpha\nu}\Gamma^\nu_{\beta\delta}V^\delta + \overline{\Gamma^\nu_{\alpha\beta}\partial_\nu V^\mu} + \cancel{\Gamma^\nu_{\alpha\beta}\Gamma^\mu_{\nu\delta}V^\delta}$$

$$= V^\nu\left(\partial_\beta\Gamma^\mu_{\alpha\nu} - \partial_\alpha\Gamma^\mu_{\alpha\nu}\right) + V^\delta\left(\Gamma^\mu_{\beta\nu}\Gamma^\nu_{\alpha\delta} - \Gamma^\mu_{\alpha\nu}\Gamma^\nu_{\beta\delta}\right)$$

$$= V^\nu\left[\partial_\beta\Gamma^\mu_{\alpha\nu} - \partial_\beta\Gamma^\mu_{\alpha\nu} + \Gamma^\mu_{\beta\delta}\Gamma^\delta_{\alpha\nu} - \Gamma^\mu_{\alpha\delta}\Gamma^\delta_{\beta\nu}\right]$$

$$= V^\nu R^\mu{}_{\nu\beta\alpha} = \boxed{R^\mu{}_{\nu\beta\alpha}V^\nu = 2\nabla_{[\beta}\nabla_{\alpha]}V^\mu = -2\nabla_{[\alpha}\nabla_{\beta]}V^\mu}$$

where

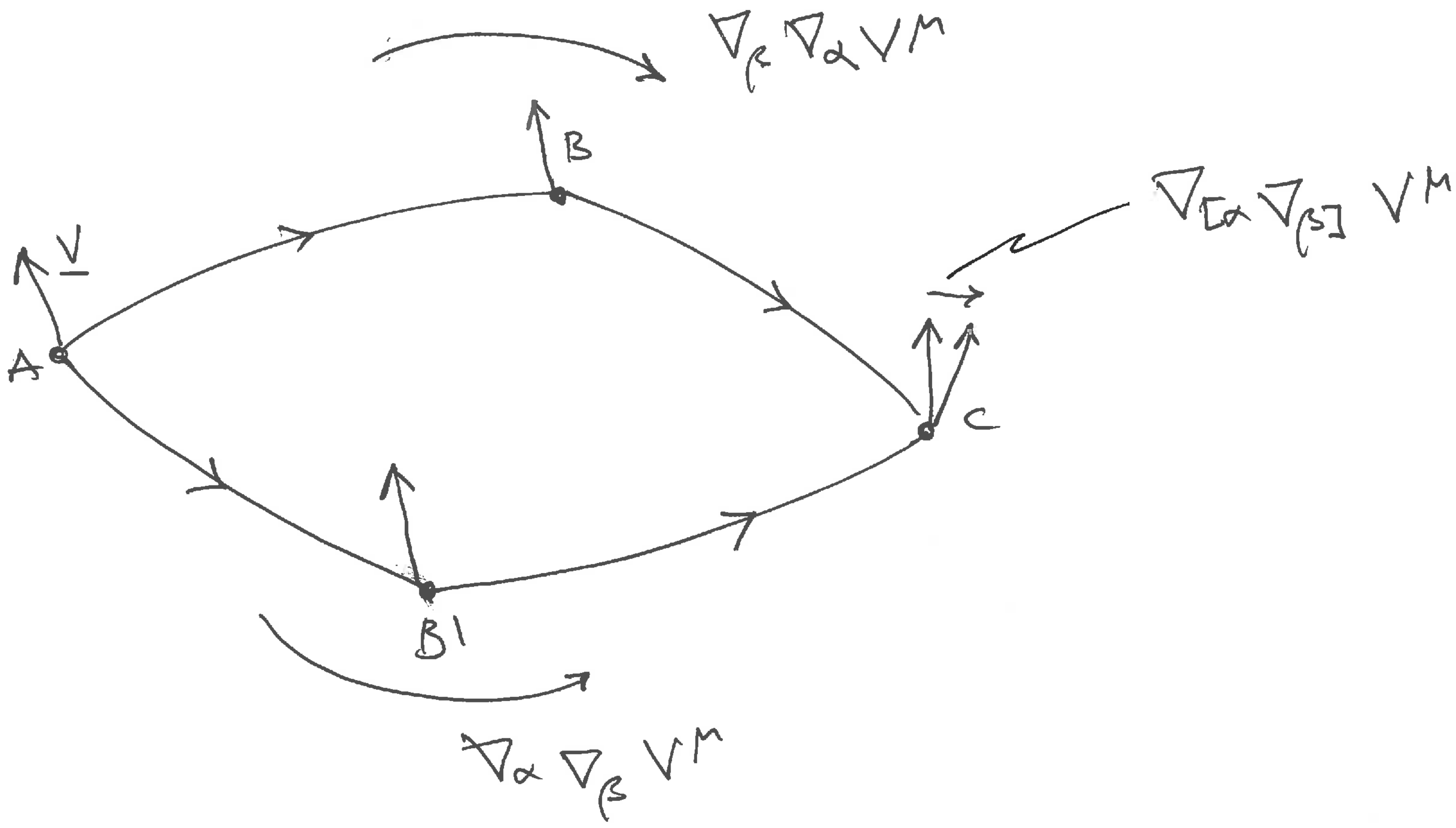
$$R^\mu{}_{\nu\beta\alpha} := \partial_\beta\Gamma^\mu_{\alpha\nu} - \partial_\beta\Gamma^\mu_{\alpha\nu} + \Gamma^\mu_{\beta\delta}\Gamma^\delta_{\alpha\nu} - \Gamma^\mu_{\alpha\delta}\Gamma^\delta_{\beta\nu}$$

Stated differently, the difference in the second covariant derivative ($\nabla_{[\alpha}\nabla_{\beta]}$) of a vector is proportional to a rank-4 tensor ($R^{\mu}{}_{\nu\beta\alpha}$) which is function of the second derivative of the metric or squares of the first derivatives of the metric, ie

$\Gamma \propto \partial g$

$$R^{\mu}{}_{\nu\beta\alpha} = R^{\mu}{}_{\nu\beta\alpha}(\partial\Gamma, \Gamma\Gamma) = R^{\mu}{}_{\nu\beta\alpha}(\partial^2 g, (\partial g)^2).$$

The operator $\nabla_{[\alpha}\nabla_{\beta]}$ when applied to a vector field also has a simple geometrical interpretation: it measures the differences in a vector when this is parallel transported in two different directions but in different order.



Using this geometrical interpretation it is not difficult to realize that $\nabla_{[\alpha} \nabla_{\beta]} V^\mu = 0$ on a plane but that $\nabla_{[\alpha} \nabla_{\beta]} V^\mu \neq 0$ on a 2-sphere.

The tensor $R^{\mu}{}_{\nu\beta\alpha}$ measures therefore the curvature and when built out of the 4D metric tensor it actually measures the curvature of spacetime.

$R^{\mu}{}_{\nu\beta\alpha}$: Riemann or curvature tensor.

- A parallely transported vector along a closed curve will not change only if the Riemann tensor is zero everywhere.
- At the same time, if the Riemann tensor is zero everywhere, the manifold is flat. Hence, only in flat manifolds a vector that is parallely transported along a closed curve will not change.

Properties of the Riemann tensor

1) $$\nabla_{[\alpha}\nabla_{\beta]} = -\nabla_{[\beta}\nabla_{\alpha]}$$

$$\Rightarrow 2\nabla_{[\alpha}\nabla_{\beta]}V^\mu = -R^\mu{}_{\nu\beta\alpha}V^\nu = -2\nabla_{[\beta}\nabla_{\alpha]}V^\mu = +R^\mu{}_{\nu\alpha\beta}V^\nu$$

$$\Leftrightarrow \quad R^\mu{}_{\nu\beta\alpha} = -R^\mu{}_{\nu\alpha\beta}$$

ie, the tensor is antisymmetric on the last two indices

2) The tensor is also antisymmetric on the first two indices (exercise)

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$$

3) The tensor is symmetric on the exchange of the first and second pair of indices.

Note

$R^{\mu}{}_{\alpha\beta\gamma} = R_{\beta\gamma}{}^{\mu}{}_{\alpha}$

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$$ (exercise)

4) The tensor is antisymmetric in the last three indices

$$R^{\mu}{}_{[\alpha\beta\delta]} = 0 \iff R^{\mu}{}_{\alpha\beta\delta} + R^{\mu}{}_{\delta\alpha\beta} + R^{\mu}{}_{\beta\delta\alpha} = 0$$

5) the tensor also satisfies an important differential identity

$$\nabla_{[\mu} R_{\nu\alpha]\beta\gamma} = 0 \iff$$

$$\nabla_{\mu} R_{\nu\alpha\beta\gamma} + \nabla_{\alpha} R_{\mu\nu\beta\gamma} + \nabla_{\nu} R_{\alpha\mu\beta\gamma} = 0$$

This is the first Bianchi identity.

6) Using the Riemann tensor and the Bianchi identity we can introduce two new tensors. In particular, we can introduce the tensor obtained from the contraction of the first and third index of the Riemann tensor

$$R^{\alpha}{}_{\beta\alpha\gamma} = R_{\beta\gamma} \qquad : \underline{\text{Ricci tensor}}$$

The Ricci tensor is symmetric

$$R_{\beta\gamma} = R_{\gamma\beta}$$

Proof

sym. on 1st and 2nd couple of indices

$$R^{\alpha}{}_{\beta\alpha\gamma} = g^{\alpha\delta} R_{\delta\beta\alpha\gamma} = g^{\alpha\delta} R_{\alpha\gamma\delta\beta}$$

$$= R^{\delta}{}_{\gamma\delta\beta} = R_{\gamma\beta} \; \square$$

7) We can further contract the two indices of the Ricci tensor and obtain the Ricci scalar

$$R := R^{\alpha}{}_{\alpha} = g^{\alpha\beta} R_{\beta\alpha}$$

Note

$$R = R^{\alpha}{}_{\alpha} = R^{\alpha\beta}{}_{\alpha\beta}$$

Proof

$$R^{\alpha}{}_{\alpha} = g^{\alpha\mu} R_{\mu\alpha} = g^{\alpha\mu} R^{\beta}{}_{\mu\beta\alpha} = R^{\beta\alpha}{}_{\beta\alpha} = R^{\alpha\beta}{}_{\alpha\beta} \quad \square$$

8) The Ricci scalar should not be confused with another scalar that can be built from the Riemann tensor, i.e. the Kretschmann scalar

$$C := R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$$

Question: what is the difference between the two tensors?

Both of them concentrate on a single number all the information on the local curvature but have different dimensions.

$$[\underline{g}] = (\text{dimensions of metric tensor}) = L^0$$

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$[ds^2] = L^2 \quad ; \quad [dx^\mu dx^\nu] = L^2 \quad \Rightarrow [g_{\mu\nu}] = L^0$$

$$[R^\mu{}_{\nu\beta\alpha}] = [\partial^2 g] = [(\partial g)^2] = L^{-2}$$

$$R = R^{\alpha\beta}{}_{\alpha\beta} \quad \Rightarrow \quad [R] = L^{-2}$$

$$G = R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} \Rightarrow [G] = L^{-2} L^{-2} = L^{-4}$$