General Relativity: Exercises

Luciano Rezzolla

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Listed below are the exercises that have been assigned during the course and collected according to the lecture in which they were assigned. These exercises can be solved independently or together during the exercise time. Some of these questions could be part of the oral exam.

Lecture I

1. Consider a binary systems of gravitating objects of mass M and m.

   - Consider first the case in which $m \ll M$ and where the small-mass object is in quasi-circular orbit around the more massive one. Draw the trajectory in two-space and the worldline in a $1+1$- and in a $2+1$-dimensional spacetime [Hint: use a coordinate system centered in M].
   - Let now $m = M$ and the binary orbit in circular orbit around the Newtonian center of mass of the system. Draw the trajectory in two-space and the worldline in a $1+1$- and in a $2+1$-dimensional spacetime [Hint: use a coordinate system centered in the Newtonian center of mass].

2. Consider a two-dimensional space and cover it with two coordinate maps: a Cartesian one $x^i = (x, y)$ and a polar one $x^{i'} = (r, \theta)$.

   - Find the coordinate transformation $\boldsymbol{f}: \ x^i \to x^{i'}$
   - Find the inverse coordinate transformation $\boldsymbol{f}^{-1}: \ x^{i'} \to x^i$
   - Find the components of the transformation matrix $\Lambda^{i'}{}_i$ and its determinant $J' := |\partial x^{i'}/\partial x^i|$.
   - Find the components of the inverse transformation matrix $\Lambda^i{}_{i'}$ and its determinant $J := |\partial x^i/\partial x^{i'}|$.
   - Show that $\Lambda^{i'}{}_i \, \Lambda^i{}_{j'} = \delta^{i'}{}_{j'}$ and that $J \, J' = 1$.

3. Consider a three-dimensional space and cover it with two coordinate maps: a Cartesian one $x^i = (x, y, z)$ and a polar one $x^{i'} = (r, \theta, \phi)$. Address all the questions in point 2.

Lecture II

1. Consider two coordinate systems in a two-dimensional space $\{x^i\} = (x, y)$ and $\{x^{i'}\} = (r, \theta)$ that are related through the well-known coordinate transformation

$$\boldsymbol{f} : \begin{cases} r = \left(x^2 + y^2\right)^{1/2} \\ \theta = \tan^{-1}(y/x) \end{cases} \tag{1}$$

and its inverse

$$\boldsymbol{f} : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \tag{2}$$

Discuss the differences between the transformation matrix employed to transform a covector in this space

$$\left(\tilde{d}x\right)_i = \Lambda^{i'}{}_i \left(\tilde{d}x\right)_{i'} , \tag{3}$$

and the one employed in the coordinate transformation

$$x^{i'} = \Lambda^{i'}{}_i \, x^i \, . \tag{4}$$

2. Consider two coordinate systems in a four-dimensional spacetime $\{x^\mu\} = (t, x, y, z)$ and $\{x^{\mu'}\} = (u, v, y, z)$ that are related through the coordinate transformation

$$\boldsymbol{f} : \begin{cases} u = t - x \\ v = t + x \end{cases} \tag{5}$$

and its inverse

$$\boldsymbol{f}^{-1} : \begin{cases} t = \frac{1}{2}(v + u) \\ \\ x = \frac{1}{2}(v - u) \end{cases} \tag{6}$$

- Compute the matrices employed in the transformations

$$x^{\mu'} = \Lambda^{\mu'}{}_\mu \, x^\mu \qquad\qquad x^\mu = \Lambda^\mu{}_{\mu'} \, x^{\mu'} \, . \tag{7}$$

- Consider a four-vector with components $U^\mu = (1, 0, 0, 0)$ in the coordinate system x^μ and compute the new components $U^{\mu'}$ in the coordinate system $x^{\mu'}$.
- Repeat the calculation for the new vector $V^\mu = (-1/2, 1/2, 0, 0)$. Interpret the results.

3. Consider a $1 + 1$ representation of the sub-spaces with two coordinate systems (t, x) and (u, v).

- Draw in the two spacetimes the worldline of a particle with velocity $\dot{x} := dx/dt = 0$.
- Draw in the two spacetimes the worldline of a particle with velocity $\dot{x} = k$ $(x = kt)$ with $k < 1$.
- Interpret the results.

Lecture III

1. Consider $\boldsymbol{T}$ as contravariant tensor of rank 2 with components $T^{\mu\nu}$. Under what conditions can this tensor be cast as the product of two contravariant vectors $\boldsymbol{U}$ and $\boldsymbol{V}$, *i.e.*, such that $T^{\mu\nu} = U^\mu V^\nu$?

2. Consider the following equation

$$T^{\mu\nu} = U^\mu + V^\nu \,. \tag{1}$$

   Is $\boldsymbol{T}$ a generic tensor?

3. Consider $\boldsymbol{F}$ as a tensor of rank 2 with covariant components $F_{\mu\nu}$ and that is antisymmetric in one coordinate system, *i.e.*, $F_{\mu\nu} = -F_{\nu\mu}$.

   - Show that $F_{\mu\nu}$ is antisymmetric in all coordinate systems.
   - Does the antisymmetry in the covariant indices apply also to the contravariant indices?
   - If so, show that $F^{\mu\nu}$ is antisymmetric in all coordinate systems.

4. Consider the antisymmetric tensor $A_{\mu\nu}$ such that $A_{\mu\nu} = -A_{\nu\mu}$ and the symmetric tensor $B^{\mu\nu}$ such that $B^{\mu\nu} = B^{\nu\mu}$. Prove the following identities:

$$A_{\mu\nu} B^{\mu\nu} = 0 \,, \tag{2}$$

$$V^{\mu\nu} A_{\mu\nu} = \frac{1}{2} \left(V^{\mu\nu} - V^{\nu\mu} \right) A_{\mu\nu} \,, \tag{3}$$

$$V^{\mu\nu} B_{\mu\nu} = \frac{1}{2} \left(V^{\mu\nu} + V^{\nu\mu} \right) B_{\mu\nu} \,, \tag{4}$$

   where $\boldsymbol{V}$ is a generic tensor of rank 2.

Lecture IV

1. Using a co-ordinate system (t, r, θ, ϕ), consider the metric line element given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] , \qquad (1)$$

where $\kappa = -1,\ 0,\ 1$.

- Show that a new co-ordinate system (t, χ, θ, ϕ) the line element (1) can be rewritten as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\chi^2 + f(\chi)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] . \qquad (2)$$

- Find the form of the function $f(\chi)$ for $\kappa = -1$, 0 and 1.
- Discuss the properties of the metric in the case of $\kappa = 0$. [Hint: two metrics $\mathbf{g}$ and $\mathbf{g}'$ are conformally related if it is possible to express them as $\mathbf{g} = \Omega\,\mathbf{g}'$, where $\Omega \equiv \Omega(x^\mu)$ is a generic function and is referred to as the *conformal factor*].

2. Using a co-ordinate system $(\eta, \chi, \theta, \phi)$, consider the metric line element given by

$$\mathrm{d}s^2 = \Omega^2\left[-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \sin^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] . \qquad (3)$$

Consider now a new co-ordinate system $(\tau, \rho, \theta, \phi)$ where

$$\tau = \frac{2\sin\eta}{\cos\chi + \cos\eta} \qquad (4)$$

$$\rho = \frac{2\sin\chi}{\cos\chi + \cos\eta} , \qquad (5)$$

and find the expression of the metric (3) in this new co-ordinate system. Discuss the properties of this new metric.

3. Given the four-vector $\mathbf{u}$ such that $u^\alpha u_\alpha = -1$ and the tensor $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$, prove the following identities

$$h_{\mu\nu}u^\mu = 0 , \qquad h^\mu{}_\nu h^\lambda{}_\mu = h^\lambda{}_\nu , \qquad h^\mu{}_\mu = 3 . \qquad (6)$$

4. Consider the following antisymmetric tensor

$$F_{\alpha\beta} = -2E_{[\alpha}u_{\beta]} + \epsilon_{\alpha\beta}{}^{\gamma\delta}H_\gamma\, u_\delta \ . \tag{7}$$

Express the vectors **E** and **H** in terms of the tensor **F**. [Hint: contract $F_{\alpha\beta}$ with u^β & $\epsilon^{\alpha\beta\gamma\delta}$ respectively.]

Lecture V

As discussed in the lecture notes, the symmetric and antisymmetric parts of a tensor are denoted respectively by round and square brackets enclosing a set of indices. In general, the symmetric part of a rank n tensor is

$$T_{(\alpha_1\ \alpha_2 \ldots \alpha_n)} = \frac{1}{n!}\,(\text{sum over all permutations of the indices } \alpha_1 \text{ to } \alpha_\mathrm{n}) \ , \quad (1)$$

and the antisymmetric part of this tensor may be written as

$$T_{[\alpha_1\ \alpha_2 \ldots \alpha_n]} = \frac{1}{n!}\,(\text{alternating sum over all permutations of the indices } \alpha_1 \text{ to } \alpha_\mathrm{n}) \ . \quad (2)$$

As an example, for the rank 3 tensor $X_{\alpha\beta\gamma}$, the symmetric and antisymmetric components are given respectively by

$$X_{(\alpha\beta\gamma)} = \frac{1}{3!}\left(X_{\alpha\beta\gamma} + X_{\beta\alpha\gamma} + X_{\gamma\alpha\beta} + X_{\alpha\gamma\beta} + X_{\beta\gamma\alpha} + X_{\gamma\beta\alpha}\right) \ , \quad (3)$$

$$X_{[\alpha\beta\gamma]} = \frac{1}{3!}\left(X_{\alpha\beta\gamma} - X_{\beta\alpha\gamma} + X_{\gamma\alpha\beta} - X_{\alpha\gamma\beta} + X_{\beta\gamma\alpha} - X_{\gamma\beta\alpha}\right) \ . \quad (4)$$

1. Let **F** be a rank-2 antisymmetric tensor, **G** a rank-2 symmetric tensor and **X** and rank-3 antisymmetric tensor. Provide explicit expressions for the following tensors: $F_{\mu\nu}$, $F_{[\mu\nu]}$, $F_{(\mu\nu)}$, $G_{[\mu\nu]}$, $G_{(\mu\nu)}$, $X_{[\alpha\beta\gamma]}$, $X_{(\alpha\beta\gamma)}$, $X_{[\alpha\beta]\gamma}$, $X_{(\alpha\beta)\gamma}$, $X_{[\alpha\beta](\gamma)}$ and $X_{(\alpha\beta)[\gamma]}$.

2. Prove the following identities:

   - $X_{((\alpha_1\ \alpha_2 \ldots \alpha_n))} = X_{(\alpha_1\ \alpha_2 \ldots \alpha_n)}$
   - $X_{[[\alpha_1\ \alpha_2 \ldots \alpha_n]]} = X_{[\alpha_1\ \alpha_2 \ldots \alpha_n]}$
   - $X_{(\alpha_1 \ldots [\alpha_l\ \alpha_m] \ldots \alpha_n)} = 0$
   - $X_{[\alpha_1 \ldots [\alpha_l\ \alpha_m] \ldots \alpha_n]} = X_{[\alpha_1 \ldots \alpha_l\ \alpha_m \ldots \alpha_n]}$

3. Let **F** be a rank-2 antisymmetric tensor with components $F^{\mu\nu}$. From **F** construct another rank-2 tensor antisymmetric tensor $^*\mathbf{F}$ such that

   $$^*\mathbf{F} := \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}\, \mathbf{e}_\mu \otimes \mathbf{e}_\nu \ . \quad (5)$$

   The tensor $^*\mathbf{F}$ is usually referred to as the *dual* of **F**. Show that the following is true

   $$^*\left(^*\mathbf{F}\right) = -\mathbf{F} \ . \quad (6)$$

4. Let $\mathbf{V}$ be a rank-3 tensor with components $V^{\alpha\beta\gamma}$ and define

$$({}^*V)^{\alpha\beta\gamma} := V_\mu \epsilon^{\mu\alpha\beta\gamma} \ . \tag{7}$$

Show that the following is true

$$V^\mu V_\mu = -\frac{1}{3!} \left({}^*V\right)^{\alpha\beta\gamma} \left({}^*V\right)_{\alpha\beta\gamma} \ . \tag{8}$$